Free Fermionic Constructions of

自由费米子构造

Heterotic Strings

杂弦

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Abstract

摘要

This chapter is an introduction to the Free Fermionic Formulation of string theory, with emphasis on heterotic model building. After a brief review of bosonization in two-dimensional conformal field theories, we discuss how internal bosonic string coordinates can be consistently replaced by free fermionic degrees of freedom. In this framework, worldsheet supersymmetry may be realized entirely among free fermions. Embedding this construction into string theory leads to a number of constraints arising from modular invariance at one and higher genera. The solution of these constraints takes the form of a small number of model building rules from which the string spectrum and interactions may be analyzed. We review some of the most well-studied models in the literature and their classification, with emphasis on the symmetric basis. The explicit map of free fermionic models to the orbifold construction is presented in some detail.

本章介绍弦理论的自由费米子表述，重点讲解杂化弦模型构建。在简要回顾二维共形场论的玻色化之后，我们讨论如何将内部玻色弦坐标一致地替换为自由费米子自由度。在该框架下，世界面超对称可以完全由自由费米子实现。将该构造嵌入弦理论后，会得到一系列由一模和高模模不变性导出的约束。这些约束的解可总结为少量模型构建规则，通过这些规则即可分析弦谱和相互作用。我们综述了文献中研究最为充分的部分模型及其分类，重点介绍对称基。文中还详细阐述了自由费米子模型到轨形构造的明确映射。

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strings

弦

Introduction

引言

In this chapter, we introduce and review the Free Fermionic Formulation (FFF) of heterotic string theory, together with a discussion of some of its main applications in the context of model building. In the FFF, all worldsheet bosonic coordinates are consistently fermionized, except for the non-compactified space-time degrees of freedom [3, 4, 118, 119]. Worldsheet supersymmetry is preserved although nonlinearly realized among the two-dimensional fermionic fields [5]. Factorization of string amplitudes, together with modular invariance constraints at one and higher genera, can be explicitly solved and give rise to a relatively simple set of rules that allow the direct construction of string models in $d \leq 10$ dimensions. Due to exact solvability of the worldsheet conformal field theory (CFT), the FFF machinery allows the complete analysis of the corresponding perturbative string spectra and their interactions.

本章中，我们将介绍并回顾杂弦理论的自由费米子表述 (FFF)，同时讨论它在模型构建领域的若干主要应用。在自由费米子表述中，除未紧致化的时空自由度外，所有世界面玻色坐标都被一致地费米化 [3, 4, 118, 119]。尽管二维费米场中的世界面超对称是非线性实现的，但它仍然得以保留 [5]。弦振幅的因式分解，以及一亏格和更高亏格的模不变性约束都可以被显式求解，由此得到一套相对简单的规则，可直接用于构建 $d \leq 10$ 维弦模型。由于世界面共形场论 (CFT) 是精确可解的，自由费米子表述这套工具可以完整分析对应的微扰弦谱及其相互作用。

Specifically in four space-time dimensions, the FFF formalism is particularly suited to the construction of string vacua with rich phenomenological characteristics. Well-studied $d = 4$ models with interesting phenomenology include the flipped $\mathrm{SU}(5) \times \mathrm{U}(1)$ model [13], the Pati-Salam model [19,132], and the Standard-like Model [92]. The classification of ten-dimensional heterotic models in the FFF has been presented in [117], while in four dimensions, it has proven very efficient for extensive scans and the classification of huge classes of heterotic string vacua [7, 78, 99, 100] .

在四维时空下，自由费米子形式体系尤其适合构建具有丰富唯象特征的弦真空。经过充分研究、拥有良好唯象性质的 $d = 4$ 维模型包括翻转 $\mathrm{SU}(5) \times \mathrm{U}(1)$ 模型 [13]、帕蒂-萨拉姆模型 [19,132] 和类标准模型 [92]。十维杂弦模型在自由费米子表述下的分类已在文献 [117] 中给出，而在四维，该表述已被证明对大范围扫描和大规模杂弦真空分类非常高效 [7, 78, 99, 100] 。

This review is organized as follows. In section ”Fermionization in Two Dimensions”, we begin with a discussion of two-dimensional worldsheet CFTs with free fermionic degrees of freedom and their equivalence to compact bosons at special radii. This equivalence is then demonstrated at the level of the one-loop string partition function. In section ”Supersymmetry Among Free Fermions”, we explain the realization of worldsheet supersymmetry entirely among the free fermionic degrees of freedom and discuss the structure of the worldsheet supercurrent. Section ”Constraints from Modular Invariance” is devoted to the study of the constraints arising from modular invariance at one and higher genera, which leads to the consistent vacuum construction rules. The latter are presented in considerable detail in section ”Model Construction Rules, Spectrum, and Effective Superpotential”, where it is shown that they reduce to a set of basis vectors encoding the fermion boundary conditions, together with a corresponding set of Generalized GSO Projections (GGSO). We illustrate the application of the construction rules in the case of a simple model, by deriving its massless spectrum and superpotential. In section ” $\mathcal{N} = 1$ Supersymmetric Models”, we review some of the most studied models with $\mathcal{N} = 1$ supersymmetry constructed in the FFF, including the flipped SU(5), Pati-Salam, and Standard Model vacua and discuss some of their phenomenological properties. Subsequently, in section ”The Symmetric Basis and Model Scans”, focusing on a symmetric basis that has been of much interest in the literature, we discuss the application of the FFF to the classification of huge sets of supersymmetric string vacua. Next, in section ”Non-supersymmetric Models”, we briefly review the construction of non-supersymmetric heterotic models in the context of the FFF and its relation to the stringy Scherk-Schwarz mechanism. Finally, in section ”Map to Orbifolds”, we present the map between a class of FFF models and their equivalent $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold counterparts, which is particularly useful for deforming the theory away from the fermionic point and addressing questions of supersymmetry breaking.

本综述结构安排如下: 在“二维费米化”一节，我们首先讨论带有自由费米子自由度的二维世界面共形场论，以及它们在特殊半径下与紧致玻色子的等价性，随后在单圈弦配分函数层面证明了该等价性。在“自由费米子中的超对称”一节，我们解释了世界面超对称如何完全由自由费米子自由度实现，并讨论了世界面超流的结构。“模不变性的约束”一节专门研究一亏格和更高亏格模不变性带来的约束，由此得到了自洽的真空构建规则。这些规则在“模型构建规则、谱和有效超势”一节有相当详细的介绍，我们展示了这些规则可归约为一组编码费米边界条件的基矢，加上对应的一组广义 GSO 投影 (GGSO)。我们通过推导一个简单模型的无质量谱和超势，演示了构建规则的应用。在“$\mathcal{N} = 1$ 维超对称模型”一节，我们回顾了在自由费米子表述下构建的、研究最多的若干 $\mathcal{N} = 1$ 维超对称模型，包括翻转 SU(5)、帕蒂-萨拉姆和标准模型真空，讨论了它们的部分唯象性质。随后在“对称基与模型扫描”一节，我们聚焦文献中广受关注的对称基，讨论了自由费米子表述在大规模超对称弦真空分类中的应用。接下来在“非超对称模型”一节，我们简要回顾了自由费米子框架下非超对称杂弦模型的构建，以及它与弦论谢尔克-施瓦茨机制的关联。最后，在“映射到 orbifold”一节，我们给出了一类自由费米子模型与其等价的 $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold 对应体之间的映射，该映射对于将理论偏离费米点形变、研究超对称破缺问题尤其有用。

Fermionization in Two Dimensions

二维费米化

The main idea behind the FFF is to exploit the bosonization property [48, 136, 148] of chiral fermions in $1+1$ dimensions in reverse and consistently replace the CFT of bosonic coordinates of string theory ascribed to the internal space with a system of auxiliary free fermions. In the two-dimensional worldsheet CFT with Minkowski signature, one may simultaneously impose both Majorana and Weyl conditions on massless fermions to reduce them to real, one-component spinors. Consider the CFT of $2N$ real, noninteracting fermions ψ^i , with $i = 1, \dots, 2N$ with action

FFF 的核心思想是反向利用 [48, 136, 148] 在 $1+1$ 维空间中手征费米子的玻色化性质，将弦理论中归属于内空间的玻色坐标共形场论一致替换为辅助自由费米子系统。在具有闵氏号差的二维世界面共形场论中，我们可以同时对无质量费米子施加马约拉纳条件外尔条件，将其约化为实单分量旋量。考虑 $2N$ 个无相互作用实费米子 ψ^i 的共形场论，其中 $i = 1, \dots, 2N$ ，作用量为

$$S = \frac{1}{2\pi} \int \mathrm{d}^2 z \left(\psi^i \bar{\partial} \psi^i + \overline{\psi}^i \partial \overline{\psi}^i \right). \tag{1}$$

The equations of motion then imply the decomposition into left-moving holomorphic spinors $\psi^i(z)$ and right-moving anti-holomorphic spinor $\overline{\psi}(\bar{z})$ and reflects the factorization of the free fermion CFT into left- and right-moving sectors, with energy-momentum tensors:

运动方程随后可分解为左行全纯旋量 $\psi^i(z)$ 和右行反全纯旋量 $\overline{\psi}(\bar{z})$ ，这体现了自由费米子共形场论可因子化为左行与右行扇区，能量动量张量为:

$$T(z) = -\frac{1}{2} :\psi^i \partial \psi^i(z): , \ \bar{T}(\bar{z}) = -\frac{1}{2} :\overline{\psi}^i \bar{\partial} \psi^i(\bar{z}): . \tag{2}$$

The conformal anomaly term in the TT and $\bar{T}\bar{T}$ operator product expansion (OPE) determines the corresponding central charges of the system as $c_L = c_R = N$. Similarly, the absence of interacting terms among the $2N$ real fermions leads to the OPEs:

TT 和 $\bar{T}\bar{T}$ 算符乘积展开 (OPE) 中的共形反常项确定系统的对应中心荷为 $c_L = c_R = N$ 。类似地，$2N$ 个实费米子之间不存在相互作用项，因此算符乘积展开为:

$$\begin{aligned}
\psi^i(z)\psi^j(w) &= \frac{\delta^{ij}}{z-w} + \dots, \\
\overline{\psi}^i(\bar{z})\overline{\psi}^j(\bar{w}) &= \frac{\delta^{ij}}{\bar{z}-\bar{w}} + \dots, \\
\psi^i(z)\overline{\psi}^j(\bar{z}) &= \text{ regular.}
\end{aligned} \tag{3}$$

The action (1) enjoys a global $\mathrm{SO}(2N)_L \times \mathrm{SO}(2N)_R$ symmetry under $\psi^i \to O_L^{ij}\psi^j$ and $\overline{\psi}^i \to O_R^{ij}\overline{\psi}^j$ with $O_L^T O_L = O_R^T O_R = 1$, generated by the (anti-)holomorphic currents

作用量 (1) 具有整体 $SO(2N)_L \times SO(2N)_R$ 对称性，在 $\psi^i \to O_L^{ij}\psi^j$ 和 $\overline{\psi}^i \to O_R^{ij}\overline{\psi}^j$ 变换下满足 $O_L^T O_L = O_R^T O_R = 1$，该对称性由 (反) 全纯流生成

$$J^{ij}(z) = \mathrm{i} : \psi^i(z)\psi^j(z) : , \ \bar{J}^{ij}(\bar{z}) = \mathrm{i} : \overline{\psi}^i(\bar{z})\overline{\psi}^j(\bar{z}) :, \tag{4}$$

for $i < j$, which are chirally conserved, $\bar{\partial}J^{ij}(z) = \partial\bar{J}^{ij}(\bar{z}) = 0$. Calculating the JJ OPE of these currents, shows that the system of $2N$ free fermions provides a level $k = 1$ realization of a Kač-Moody current algebra ${}^1SO(2N)_L \times SO(2N)_R$.

对于 $i < j$，这些流是手征守恒的，$\bar{\partial}J^{ij}(z) = \partial\bar{J}^{ij}(\bar{z}) = 0$。计算这些流的 JJ 算符乘积展开可知，$2N$ 个自由费米子构成的系统给出了卡茨-穆迪流代数 ${}^1SO(2N)_L \times SO(2N)_R$ 的水平 $k = 1$ 实现。

We now focus on the left-movers, since the same analysis can be straightforwardly performed on the right-moving CFT as well. It is convenient to complexify the real fermions into pairs:

我们现在聚焦左行部分，因为同样的分析也可以直接应用到右行共形场论。将实费米子复化为配对形式更为方便:

$$\Psi^{a,\pm}(z) = \frac{1}{\sqrt{2}}\left(\psi^{2a}(z) \pm \mathrm{i}\psi^{2a-1}(z)\right), \tag{5}$$

where $a = 1, \ldots, N$ runs over the rank of the $SO(2N)$ symmetry and similarly for the right-movers. It is possible to bosonize the left-moving CFT of the N complexified fermions $\Psi^{a,\pm}(z)$, that is, to consistently replace it with a system of compact scalar fields $H^a(z)$. Normalizing their two-point correlators on the sphere as

其中 $a = 1, \ldots, N$ 遍历 $SO(2N)$ 对称性的秩，右行部分同理。我们可以对 N 个复化费米子 $\Psi^{a,\pm}(z)$ 的左行共形场论进行玻色化，也就是将其一致替换为紧化标量场 $H^a(z)$ 系统。将它们在球面上的两点关联函数归一化为

$$\left\langle H^a(z) H^b(w) \right\rangle = -\frac{\alpha'}{2}\delta^{ab}\log(z-w), \tag{6}$$

and adopting for the time being the CFT convention $\alpha' = 2$, the chiral bosonization corresponds to the identification [2]

暂时采用共形场论约定 $\alpha' = 2$，手征玻色化对应于识别 [2]

$$\Psi^{a,\pm}(z) =: \mathrm{e}^{\pm iH^a(z)} : , \ J^a(z) = \mathrm{i}\partial H^a(z) =: \Psi^{a,+}\Psi^{a,-} : (z), \tag{7}$$

and we henceforth suppress the explicit display of the normal ordering symbols. It is straightforward to check that the $J^a(z)$ currents defined in (7) correspond to the N Cartan currents of the Kač-Moody algebra $SO(2N)$, while the remaining $4\begin{pmatrix} N \\ 2 \end{pmatrix}$ currents can be obtained as the combinations $\mathrm{e}^{\pm \mathrm{i}H^a(z) \pm \mathrm{i}H^b(z)}$. Furthermore, it can be checked that the bosonization (7) is consistent with the conformal weights, reproduces the same OPEs as the original fermion system, and is therefore equivalent to the CFT of the N free compact scalars H with the same left-moving central charge $c_L = N$.

因此我们接下来不再显式写出正规序符号。可以直接验证，式 (7) 定义的 $J^a(z)$ 流对应卡-穆迪代数 $\mathrm{SO}(2N)$ 的 N 嘉当流，其余的 $4\begin{pmatrix} N \\ 2 \end{pmatrix}$ 流可通过组合 $\mathrm{e}^{\pm \mathrm{i} H^a(z) \pm \mathrm{i} H^b(z)}$ 得到。此外还可验证，玻色化 (7) 与共形权重一致，能重现出和原费米子系统完全相同的算子乘积展开，因此它等价于具有相同左行中心荷 $c_L = N$ 的 N 自由紧致标量 H 的共形场论。

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[1] An excellent review of affine current algebras can be found in [109].

[1] 仿射流代数的优秀综述可见文献 [109]。

[2] Although the precise relation between Ψ and H is complicated and nonlocal [136], a simple method of bosonization [88] can be obtained, by introducing appropriate cocycle phases that ensure the correct fermion anti-commutation relations. While we suppress cocycles in this presentation, an excellent discussion of the technique, particularly useful for concrete calculations, can be found in [113].

[2] 尽管 Ψ 与 H 之间的精确关系复杂且非局域 [136]，但我们可以通过引入合适余循环相位来保证正确的费米子反对易关系，从而得到一种简单的玻色化方法 [88]。本文在讨论中省略了余循环，该技术非常适合具体计算，相关的优秀讨论可见文献 [113]。

The (radial) quantization of the free fermion system, of course, depends on the boundary conditions along the nontrivial cycle of the cylinder, which can be Neveu-Schwarz (antiperiodic) or Ramond (periodic). In particular, the Ramond vacuum is degenerate and transforms as an $\mathrm{SO}(2N)$ spinor, whose irreducible Weyl representations of opposite chirality will be denoted as S (spinor) and C (conjugate spinor). The associated spin fields $S(z)$ and $C(z)$ are then constructed such that they generate the Ramond vacuum, upon acting on the $\mathrm{SL}(2;\mathbb{C})$ invariant vacuum.

当然，自由费米子系统的 (径向) 量子化依赖于圆柱非平凡闭链上的边界条件，边界条件可以是诺伊沃-施瓦茨 (反周期) 边界条件，也可以是拉蒙德 (周期) 边界条件。特别地，拉蒙德真空是简并的，按 $\mathrm{SO}(2N)$ 旋量变换，其相反手性的不可约外尔表示记为 S (旋量) 和 C (共轭旋量)。相应的旋量场 $S(z)$ 和 $C(z)$ 构造为: 作用在 $\mathrm{SL}(2;\mathbb{C})$ 不变真空上即可生成拉蒙德真空。

These spin fields admit a simple free-field representation in terms of the bosonization fields H . Indeed, both the $\psi(z)S(0)$ and $\psi(z)C(0)$ OPEs involve branch cuts in z which identify the helicity charges q_a, q'_a of the spin fields as $q_a, q'_a \in \left\{\pm\frac{1}{2}\right\}$

这些旋量场可以用玻色化场 H 表示为简单的自由场形式。事实上，$\psi(z)S(0)$ 和 $\psi(z)C(0)$ 的算子乘积展开在 z 中都存在分支切割，这说明旋量场的螺旋度荷 q_a, q'_a 为 $q_a, q'_a \in \left\{\pm\frac{1}{2}\right\}$

$$S(z) = \mathrm{e}^{\mathrm{i} q_a H^a(z)},\ C(z) = \mathrm{e}^{\mathrm{i} q'_a H^a(z)}. \tag{8}$$

In this helicity basis, the representation is encoded into the choice of weight vectors q_a and q'_a , and the Weyl condition requires keeping an even (odd) number of minus signs in the helicity charges q_a and an odd (even) one in q'_a . Note, furthermore, that the conformal weights of $S(z)$ and $C(z)$ are obtained as the squared lengths of the weight vectors and yield $q \cdot q/2 = q' \cdot q'/2 = N/8$. In particular, for $N = 1$, this implies that the Ramond vacuum of a single complex fermion carries conformal weight $\frac{1}{8}$, in accordance with the fact that the $c = \frac{1}{2}$ Virasoro algebra contains two irreducible representations of highest conformal weight $\frac{1}{16}$, corresponding to the ground state of a single real fermion with periodic boundary conditions.

在该螺旋度基下，表示由权向量 q_a 和 q'_a 的选择确定，外尔条件要求在 q_a 的螺旋度荷中保留偶数 (奇数) 个负号，在 q'_a 中保留奇数 (偶数) 个负号。此外注意，$S(z)$ 和 $C(z)$ 的共形权重等于权向量的长度平方，结果为 $q \cdot q/2 = q' \cdot q'/2 = N/8$ 。特别地，对 $N = 1$ 而言，这说明单个复费米子的拉蒙德真空的共形权重为 $\frac{1}{8}$ ，这与如下事实一致: $c = \frac{1}{2}$ 维拉索罗代数包含两个最高共形权重为 $\frac{1}{16}$ 的不可约表示，对应单个实费米子在周期边界条件下的基态。

The mode expansion of a free non-compact scalar $\Phi(z, \bar{z})$ carries continuous momentum modes $p_L = p_R$ which are always left-right matched, hence making it incompatible with the holomorphic factorization of the free fermion CFT. It is then clear that the chiral bosons $H^a(z)$ must be necessarily compact in order to consistently bosonize the free fermion system. Consider for simplicity a single chiral boson $H(z)$, compactified on a circle of radius R . Reinstating the α' -dependence in (7)

自由非紧致标量 $\Phi(z, \bar{z})$ 的模展开包含连续动量模 $p_L = p_R$ ，这些动量模始终满足左右匹配，因此它与自由费米子共形场论的全纯分解不兼容。由此可见，为了一致地玻色化自由费米子系统，手征玻色子 $H^a(z)$ 必须是紧致的。为简化起见，我们考虑单个紧致在半径为 R 的圆上的手征玻色子 $H(z)$ ，将 α' 依赖关系重新引入式 (7)

$$\Psi^{\pm}(z) = \mathrm{e}^{\pm \mathrm{i}\sqrt{\frac{2}{\alpha'}}H(z)}, \; J(z) = \mathrm{i}\sqrt{\frac{2}{\alpha'}}\partial H(z) =: \Psi^+\Psi^- :(z), \tag{9}$$

and requiring that the bosonized representation for $\Psi^{\pm}(z)$ be single valued under $H \to H + 2\pi R$, we obtain the $R = \sqrt{\alpha'/2}$. This implies that the fermionization of string coordinates is only consistent at specific points in moduli space, known as the fermionic point.

并要求 $\Psi^{\pm}(z)$ 的玻色化表示在 $H \to H + 2\pi R$ 下是单值的，我们便得到 $R = \sqrt{\alpha'/2}$ 。这意味着弦坐标的费米化仅在模空间中被称为费米点的特殊位置上是自洽的。

We are now ready to describe the bosonization procedure at the level of the string partition function. At genus 1, we must specify the boundary conditions of fields along both nontrivial cycles of the worldsheet torus $z \sim z + 1 \sim z + \tau$ of complex structure $\tau = \tau_1 + \mathrm{i}\tau_2$. Consider the chiral complexified fermion $\Psi^{\pm}(z)$ with boundary conditions

现在我们可以在弦配分函数的层面描述玻色化过程。在亏格 1 的情况下，我们必须指定场在世界面环面 $z \sim z + 1 \sim z + \tau$ 的两个非平凡闭链上的边界条件，该环面的复结构为 $\tau = \tau_1 + \mathrm{i}\tau_2$ 。考虑带有如下边界条件的手征复化费米子 $\Psi^{\pm}(z)$

$$\Psi^{\pm}(z+1) = -\mathrm{e}^{\mp \mathrm{i}\pi\gamma}\Psi^{\pm}(z),$$

$$\Psi^{\pm}(z+\tau) = -\mathrm{e}^{\pm \mathrm{i}\pi\delta}\Psi^{\pm}(z)\,, \tag{10}$$

where γ, δ are real parameters twisting the boundary conditions. Note that the special values $\gamma = 0, 1$ correspond to antiperiodic and periodic boundary conditions along the a -cycle of the torus, respectively. Similarly, $\delta = 0, 1$ assign antiperiodic and periodic boundary conditions along the b -cycle. Of course, for a single real fermion $\psi(z)$, the assignments $\gamma, \delta = 0, 1$ would be the only distinct boundary conditions allowed by the $\mathbb{Z}_2$ automorphism $\psi \to -\psi$ of the left-moving CFT. In the case of a complexified fermion $\Psi^{\pm}(z)$, the CFT now enjoys a continuous $\mathrm{U}(1)$ symmetry, which allows for arbitrary twistings γ, δ of the boundary conditions.

其中 γ, δ 是扭转边界条件的实参数。注意特殊值 $\gamma = 0, 1$ 分别对应环面 a 闭链上的反周期和周期边界条件；同理，$\delta = 0, 1$ 对应 b 闭链上的反周期和周期边界条件。当然，对于单个实费米子 $\psi(z)$ ，由左行经共形场论的 $\mathbb{Z}_2$ 自同构 $\psi \to -\psi$ 所允许的不同边界条件只有 $\gamma, \delta = 0, 1$ 。在复化费米子 $\Psi^{\pm}(z)$ 的情况下，共形场论具有连续的 $\mathrm{U}(1)$ 对称性，允许边界条件取任意扭转 γ, δ 。

The fermionic path integral essentially amounts to the evaluation of the determinant of a chiral Dirac operator $\overline{\partial}$ twisted by γ, δ . Up to an irrelevant phase, this reads

费米子路径积分本质上等价于计算被 γ, δ 扭转的手征狄拉克算符 $\overline{\partial}$ 的行列式。忽略不影响结果的相位后，结果为

$$\mathrm{Det}_{\gamma,\delta}\left(\overline{\partial}\right) = \mathrm{e}^{\mathrm{i}\pi\frac{\gamma\delta}{2}} q^{\frac{\gamma^2}{8}-\frac{1}{24}} \prod_{n>0}\left(1+q^{n+\frac{\gamma}{2}-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\pi\delta}\right)\left(1-q^{n-\frac{\gamma}{2}-\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\pi\delta}\right)$$

(11)

$$= \frac{\vartheta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}(0;\tau)}{\eta(\tau)},$$

and matches the expression for the partition function of a complex chiral fermion obtained by canonical quantization. [3] Here, η is the Dedekind function:

与正则量子化得到的复手征费米子配分函数表达式一致。[3] 此处 η 是戴德金函数:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n>0}(1-q^n)\,, \tag{12}$$

expressed as a function of the nome $q \equiv \mathrm{e}^{2\pi\mathrm{i}\tau}$, while $\vartheta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}(z;\tau)$ are the Jacobi theta functions with characteristics, with sum representation:

其用 nome $q \equiv \mathrm{e}^{2\pi\mathrm{i}\tau}$ 表示，而 $\vartheta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}(z;\tau)$ 是带特征的雅可比 θ 函数，求和表示为:

$$\vartheta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}(z;\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n-\frac{\gamma}{2}\right)^2} \mathrm{e}^{2\pi\mathrm{i}\left(z-\frac{\delta}{2}\right)\left(n-\frac{\gamma}{2}\right)}. \tag{13}$$

For simplicity, theta constants $\vartheta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}(0;\tau)$ will be henceforth denoted simply as $\vartheta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}$.

为简化起见，此后我们将 θ 常数 $\vartheta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}(0;\tau)$ 简记为 $\vartheta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}$.

Now consider the CFT of a single complex fermion $\Psi^{\pm}(z)$ with boundary conditions $\begin{bmatrix}\gamma\\ \delta\end{bmatrix}$. Together with its right-moving counterpart $\overline{\Psi}^{\pm}(\bar{z})$ and assuming the same boundary conditions, the contribution to the partition function reads

现在考虑带有边界条件 $\begin{bmatrix}\gamma\\ \delta\end{bmatrix}$ 的单个复费米子 $\Psi^{\pm}(z)$ 的共形场论。加上其右行对应物 $\overline{\Psi}^{\pm}(\bar{z})$ 并假定边界条件相同，它对配分函数的贡献为

$$Z_{\Psi\overline{\Psi}}\begin{bmatrix}\gamma\\ \delta\end{bmatrix}=\frac{\vartheta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}\bar{\vartheta}\begin{bmatrix}\gamma\\ \delta\end{bmatrix}}{\eta\bar{\eta}}=\frac{1}{\eta\bar{\eta}}\sum_{m,n\in\mathbb{Z}}q^{\frac{1}{2}\left(m-\frac{\gamma}{2}\right)^2}\bar{q}^{\frac{1}{2}\left(n-\frac{\gamma}{2}\right)^2}\mathrm{e}^{-\mathrm{i}\pi\delta(m-n)} \tag{14}$$

where we explicitly made use of the sum representation (13) of theta constants. Shifting the summation variable $n\to m-n$, we can rearrange the exponents as

其中我们明确利用了西塔常数的求和表示式 (13)。对求和变量 $n\to m-n$ 做平移后，我们可将指数重新整理为

$$Z_{\Psi\overline{\Psi}}\begin{bmatrix}\gamma\\ \delta\end{bmatrix}=\frac{1}{\eta\bar{\eta}}\sum_{m,n\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}\pi n\delta}q^{\frac{\alpha'}{4}\left(\frac{m-n/2-\gamma/2}{\sqrt{\alpha'/2}}+\frac{n}{\sqrt{2\alpha'}}\right)^2}\bar{q}^{\frac{\alpha'}{4}\left(\frac{m-n/2-\gamma/2}{\sqrt{\alpha'/2}}-\frac{n}{\sqrt{2\alpha'}}\right)^2},$$

(15)

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[3] Our conventions are a modified version of [18]. See the same reference for a detailed discussion of the properties of chiral Dirac determinants on Riemann surfaces.

[3] 我们的约定是文献 [18] 的修正版本。关于黎曼曲面上手征狄拉克行列式的性质的详细讨论参见该文献。

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and we recognize on the r.h.s. the shifted (1, 1) lattice

我们可以在等式右侧识别出平移后的 (1, 1) 晶格

$$\hat{\Gamma}_{1,1}\begin{bmatrix}\gamma\\ \delta\end{bmatrix}(R)=\sum_{m,n\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}\pi n\delta}q^{\frac{\alpha'}{4}\left(\frac{m-n/2-\gamma/2}{R}+\frac{nR}{\alpha'}\right)^2}\bar{q}^{\frac{\alpha'}{4}\left(\frac{m-n/2-\gamma/2}{R}-\frac{nR}{\alpha'}\right)^2}, \tag{16}$$

at the fermionic radius $R = \sqrt{\alpha'/2}$. Summing over all spin structures $\begin{bmatrix} \gamma \\ \delta \end{bmatrix}$ in (15), we recover the partition function of a scalar compactified on a circle S^1 of the same radius. Indeed, the sum over γ effectively resets $2m - \gamma \to m$ to integer values, while the sum over δ projects onto even windings $n \to 2n$,

对应费米半径 $R = \sqrt{\alpha'/2}$。对 (15) 式中所有自旋结构 $\begin{bmatrix} \gamma \\ \delta \end{bmatrix}$ 求和后，我们得到了相同半径下紧致化在圆 S^1 上的标量的配分函数。事实上，对 γ 的求和实际上将 $2m - \gamma \to m$ 重置为整数值，而对 δ 的求和则投影到偶数缠绕 $n \to 2n$,

$$\frac{1}{2} \sum_{\gamma,\delta\in\mathbb{Z}_2} Z_{\Psi\overline{\Psi}} \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \frac{1}{\eta\overline{\eta}} \sum_{m,n\in\mathbb{Z}} q^{\frac{\alpha'}{4}\left(\frac{m}{\sqrt{\alpha'/2}}+\frac{n}{\sqrt{2\alpha'}}\right)^2} \bar{q}^{\frac{\alpha'}{4}\left(\frac{m}{\sqrt{\alpha'/2}}-\frac{n}{\sqrt{2\alpha'}}\right)^2} = \frac{\Gamma_{1,1}\left(\sqrt{\frac{\alpha'}{2}}\right)}{\eta\overline{\eta}}, \tag{17}$$

where, again, on the r.h.s., we recognize the (unshifted) Narain lattice [137, 138]

在这里，我们同样可以在等式右侧识别出 (未平移的)Narain 晶格 [137, 138]

$$\Gamma_{1,1}(R) = \sum_{m,n\in\mathbb{Z}} q^{\frac{\alpha'}{4}P_L^2} \bar{q}^{\frac{\alpha'}{4}P_R^2}, \tag{18}$$

at the fermionic radius, expressed in terms of the left- and right-moving compact momenta $P_{L,R} = \frac{m}{R} \pm \frac{nR}{\alpha'}$, with $m, n \in \mathbb{Z}$ denoting the Kaluza-Klein momentum and winding numbers, respectively. Analogous expressions can be obtained in the case of several complexified fermions, giving rise to higher-dimensional lattices. We will return to this point when we discuss the connection of the FFF to orbifold theories in section ”Map to Orbifolds”.

它对应费米半径，用左行和右行紧致动量 $P_{L,R} = \frac{m}{R} \pm \frac{nR}{\alpha'}$ 表示，其中 $m, n \in \mathbb{Z}$ 分别对应卡鲁扎-克莱因动量和缠绕数。对于多个复化费米子可以得到类似的表达式，它们对应更高维的晶格。我们会在“映射到轨形”一节讨论 FFF 与轨形理论的联系时回到这一点。

Supersymmetry Among Free Fermions

自由费米子中的超对称

The equivalence between the worldsheet CFTs of free fermions and free scalars compactified at the fermionic radius $R = \sqrt{\alpha'/2}$ opens up the possibility of constructing consistent string theories in which all worldsheet (internal) degrees of freedom are consistently fermionized at special points in moduli space. Embedding this procedure in the string worldsheet is, however, far from trivial. On the one hand, in the case of the superstring, the fermionic degrees of freedom we introduce must realize a local $\mathcal{N} = 1$ superconformal algebra, necessary for the consistent coupling of the theory to two-dimensional gravity and for projecting out unphysical states. On the other hand, not all choices of boundary conditions for the free fermions are consistent with modular invariance at one and higher genera. In this section, we will begin discussing these requirements and the constraints they imply. [4]

自由费米子与紧致化在费米半径 $R=\sqrt{\alpha'/2}$ 下的自由标量世界面 CFT 之间的等价性，为构造自洽弦理论开辟了可能: 这类理论中，所有世界面 (内) 自由度都可以在模空间的特殊点上被一致费米化。然而，将该方法嵌入弦世界面绝非易事。一方面，对于超弦而言，我们引入的费米自由度必须实现定域 $\mathcal{N}=1$ 超共形代数——这是理论与二维引力自洽耦合，以及投影掉非物理态所必需的。另一方面，并非自由费米所有边界条件的选择都能保证亏格为 1 及更高亏格下的模不变性。在本节中，我们将开始讨论这些要求及其带来的约束。[4]

Consider a theory of N free left-moving Majorana-Weyl fermions ψ^i with action

考虑 N 个自由左行 Majorana-Weyl 费米子 ψ^i 的理论，其作用量为

$$S=\frac{1}{2\pi}\int \mathrm{d}^2 z\psi^i\bar{\partial}\psi^i, \tag{19}$$

enjoying a global $\mathrm{O}(N)$ symmetry, with $i=1,2,\dots,N$. Observe that the action is invariant under the nonlinear supersymmetry transformation

具有整体 $\mathrm{O}(N)$ 对称性，满足 $i=1,2,\dots,N$ 。可以看到，该作用量在非线性超对称变换下保持不变

$$\delta\psi^i=\varepsilon C^{ijk}\psi^j\psi^k, \tag{20}$$

if and only if C^{ijk} is fully antisymmetric [55,108]. Constructing the generator of this transformation leads to the fermionic realization of the worldsheet supercurrent

当且仅当 C^{ijk} 是全反对称的 [55,108]。构造该变换的生成元可以得到世界面超流的费米实现

$$T_F(z)=\frac{1}{3}\mathrm{i}C^{ijk}\psi^i\psi^j\psi^k(z), \tag{21}$$

which indeed carries the correct $\left(0,\frac{3}{2}\right)$ conformal weight. We should now impose that the $T_F(z)$ in (21) indeed closes an $\mathcal{N}=1$ superconformal algebra, i.e., that it satisfies the OPE

它确实具有正确的 $\left(0,\frac{3}{2}\right)$ 共形权重。我们现在需要证明，式 (21) 中的 $T_F(z)$ 确实闭合一个 $\mathcal{N}=1$ 超共形代数，即它满足算符乘积展开 (OPE)

$$T_F(z)T_F(w)=\frac{\hat{c}}{(z-w)^3}+\frac{2}{z-w}T(z)+\dots, \tag{22}$$

where $\hat{c}=2c/3$, with $c=N/2$ being the central charge of the free fermion CFT. Calculating the same OPE using the explicit form of the supercurrent (21), one obtains

其中 $\hat{c}=2c/3$ ， $c=N/2$ 是自由费米子 CFT 的中心荷。利用超流 (21) 的显式形式计算同一个 OPE，可以得到

$$T_F(z)T_F(w)=\frac{2}{3}\frac{C^{ijk}C^{ijk}}{(z-w)^3}+2C^{ijk}C^{ij\ell}\frac{:\psi^k\psi^\ell:(w)}{(z-w)^2}$$

$$-2C^{ijk}C^{ij\ell}\frac{:\psi^k\partial\psi^\ell:(w)}{z-w}-C^{ijk}C^{imn}\frac{:\psi^j\psi^k\psi^m\psi:^n(w)}{z-w}+\ldots$$

(23)

The requirement that this matches (22) implies two independent conditions [5]:

要求该结果与式 (22) 一致会给出两个独立条件 [5]:

$$C^{[ij]m}C^{k\ell]m}=0, \tag{24}$$

$$C^{ijk}C^{ij\ell}=\frac{1}{2}\delta^{k\ell}. \tag{25}$$

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4 For more details, see [5].

4 更多细节参见文献 [5]。

The first is a Jacobi identity, implying that C^{ijk} are structure constants of a Lie algebra corresponding to a group G , whereas the second implies that G be semisimple and compact. In other words, requiring the realization of an $\mathcal{N}=1$ superconformal field theory implies that the N real fermions ψ^i must transform in the adjoint representation of a Lie group, such that the global $\mathrm{SO}(N)$ symmetry is gauged into a local symmetry G , satisfying $\dim G=N$. We will henceforth write the structure constants in the conventional normalization

第一个条件是雅可比恒等式，意味着 C^{ijk} 是对应于群 G 的李代数结构常数，第二个条件则要求 G 是半单紧致群。换言之，要求实现一个 $\mathcal{N}=1$ 超共形场论意味着 N 个实费米子 ψ^i 必须在李群的伴随表示下变换，从而整体 $\mathrm{SO}(N)$ 对称性被规范化为定域对称性 G ，满足 $\dim G=N$ 。下文中我们将采用常规归一化来表示结构常数:

$$C^{ijk}=\frac{1}{2\sqrt{h^\vee}}f^{ijk} \tag{26}$$

where $h^\vee=f^{ijk}f^{ijk}/2\dim(G)$ is the dual Coxeter number [5] of G .

其中 $h^\vee=f^{ijk}f^{ijk}/2\dim(G)$ 是 G 的对偶 Coxeter 数 [5] 。

Consider now the left-moving sector of superstring theory which, as mentioned above, must enjoy at least $\mathcal{N}=1$ worldsheet supersymmetry. Denoting the number of non-compact dimensions as D , and assuming that the internal space is realized entirely in terms of N real free fermions with $\mathrm{SO}(N)$ global symmetry, it is clear that the cancellation of the conformal anomaly requires the vanishing of the total central charge

现在考虑超弦理论的左行 sector，正如上文所述，它至少需要具备 $\mathcal{N}=1$ 个世界面超对称。将非紧致维度的数量记为 D ，假设内部空间完全由带有 $\mathrm{SO}(N)$ 整体对称性的 N 个实自由费米子描述，显然共形反常抵消要求总中心荷为零:

$$\frac{3}{2}D + N/2 - 15 = 0, \tag{27}$$

with $3D/2$ being the contribution of the D non-compact super-coordinates, $N/2$ that of the system of auxiliary real fermions, and -15 being the net contribution of the b, c, β, γ (super)ghost systems.

其中 $3D/2$ 是 D 个非紧致超坐标的贡献，$N/2$ 是辅助实费米子系统的贡献，-15 是 b, c, β, γ (超) 鬼系统的净贡献。

Note that, although it is in many cases possible (and useful) to use bosoniza-tion in order to reinterpret the system of N auxiliary fermions in terms of (super-)coordinates compactified at the fermionic radius, the idea of the fermionic construction is that it is possible to directly construct consistent string theories in $D < 10$ space-time dimensions, by balancing the central charge deficit created by the lack of a traditional compactification space, against the system of $N = 3\,(10 - D)$ worldsheet fermions. Moreover, we shall see that doing so allows for a complete classification of string models constructible in the free fermionic framework in terms of simple constraints for the boundary conditions of the fermion system and its associated generalized Gliozzi-Scherk-Olive (GSO) projections.

请注意，尽管在很多情况下我们可以 (也确实有用) 使用玻色化，在费米半径紧致化的 (超) 坐标下重释 N 辅助费米子系统，但费米构造的核心思想是: 可以在 $D < 10$ 时空维数中直接构造自洽的弦理论，方法是平衡传统紧致化空间缺失带来的中心荷亏缺与 $N = 3\,(10 - D)$ 世界面费米子系统。此外我们将会看到，通过该构造，我们可以基于费米系统边界条件及其关联的广义格里奥齐-谢克-奥利夫 (GSO) 投影的简单约束，完成自由费米框架下可构造弦模型的完全分类。

Specializing to $D = 4$ dimensions, we see immediately that a total of $N = 18$ auxiliary real fermions need to be introduced into the left-moving CFT. If all 18 real fermions share the same boundary conditions, the unbroken global symmetry group is SO(18), and this has to be gauged down to the compact, semi-simple gauge group G . The only such groups of dimension 18 are $\mathrm{SU}(2)^6$, $\mathrm{SU}\,(3) \times \mathrm{SO}\,(5)$, and $\mathrm{SU}\,(4) \times \mathrm{SU}\,(2)$, each realizing a super-Kač-Moody current algebra at level $k = h^\vee$ whose JJ OPE reads

特殊化到 $D = 4$ 维后，我们可以立即看出，需要向左行 CFT 引入总共 $N = 18$ 个辅助实费米子。若全部 18 个实费米子具有相同边界条件，未破缺的整体对称群为 SO(18)，必须将其规范约化为紧致半单规范群 G 。满足维数为 18 的这类群只有 $\mathrm{SU}(2)^6$, $\mathrm{SU}\,(3) \times \mathrm{SO}\,(5)$ 和 $\mathrm{SU}\,(4) \times \mathrm{SU}\,(2)$ ，二者都实现了 level $k = h^\vee$ 的超卡-穆迪流代数，其 JJ OPE 为

$$J^i\,(z)\,J^j\,(w) = \frac{k\delta^{ab}}{(z-w)^2} + \mathrm{i} f^{ijk}\frac{J^k\,(w)}{z-w} + \ldots \tag{28}$$

---

[5] Note the relation $c\,(G) = 2h^\vee$, with $c\,(G)$ being the quadratic Casimir in the adjoint representation of G .

[5] 注意关系 $c\,(G) = 2h^\vee$ ，其中 $c\,(G)$ 是 G 伴随表示的二次卡西米尔量。

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with the correct central charge

具有正确的中心荷

$$c = \frac{k\,\mathrm{dim}\,(G)}{k + h^{\vee}} = \frac{N}{2}. \tag{29}$$

This closure of the $\mathcal{N}=1$ superconformal algebra can be explicitly verified by taking the currents $J^i(z) = \frac{1}{2} f^{ijk}\psi^j\psi^k(z)$ as the superpartners of the free fermions ψ^i . For instance, in the maximal rank case $\mathrm{SU}(2)^6$, we have a $k=2$ realization of the current algebra, and the corresponding central charge $c = 6 \times 3/2$ precisely reflects a system of three free fermions for each SU(2) factor.

$\mathcal{N}=1$ 超共形代数的这种封闭性可以通过验证: 将流 $J^i(z) = \frac{1}{2} f^{ijk}\psi^j\psi^k(z)$ 取为自由费米子 ψ^i 的超伙伴即可显式验证。例如，在最大秩情形 $\mathrm{SU}(2)^6$ 中，我们得到流代数的 $k=2$ 实现，对应的中心荷 $c = 6 \times 3/2$ 准确反映了每个 SU(2) 因子对应三个自由费米子的系统。

It is well known [28, 94, 105] that in order for the left (resp. right)-moving sector of string theory to include massless space-time fermions, the corresponding left (right) moving super-Kač-Moody algebra must necessarily be abelian. For instance, in type II theories with unbroken space-time supersymmetry, both the left- and right-moving current algebras are abelian, and hence, non-abelian interactions may arise only by introducing D-branes. In heterotic theories, instead, only the left-moving sector enjoys worldsheet supersymmetry, and thus, it is only the left-moving current algebra that is constrained to be abelian. Indeed, the right-moving worldsheet is bosonic, and its current algebra can give rise to non-abelian currents, which translate to the presence of non-abelian gauge fields in the massless spectrum, while the spin fields making up the vertex operator of massless gravitini arise from the left-moving sector. In what follows, we focus the analysis entirely on the heterotic string.

众所周知 [28, 94, 105] ，若要弦理论的左行 (对应右行) sector 包含无质量时空费米子，对应的左行 (右行) 超卡-穆迪代数必须是阿贝尔的。例如，在具有未破缺时空超对称的 II 型弦理论中，左行与右行流代数都是阿贝尔的，因此非阿贝尔相互作用只能通过引入 D 膜产生。而在杂弦理论中，只有左行 sector 具备世界面超对称，因此仅左行流代数被约束为阿贝尔的: 右行世界面确实是玻色性的，其流代数可以产生非阿贝尔流，对应无质量谱中存在非阿贝尔规范场，而构成无质量引力微子顶点算子的旋量场则来自左行 sector。在下文中，我们将分析完全聚焦于杂弦。

We will now pick the maximal rank case, with local gauge symmetry $\mathrm{SU}(2)^6$, and further gauge it down to its abelian factors $\mathrm{U}(1)^6$, by an appropriate choice of boundary conditions on the free fermions. Such gaugings down to a subgroup H are consistent, provided G/H is a symmetric space [5]. Consider each SU(2) triplet of fermions $\{\chi^I, y^I, \omega^I\}$ with $I = 1, \ldots, 6$, such that the left-moving worldsheet supercurrent takes the form

我们现在选取极大秩情形，其局域规范对称性为 $\mathrm{SU}(2)^6$ ，再通过对自由费米子选取合适的边界条件将其规范约化到阿贝尔因子 $\mathrm{U}(1)^6$ 。只要 G/H 是对称空间，这种向下约化到子群 H 的规范操作就是自洽的 [5]。考虑每一组费米子 $\{\chi^I, y^I, \omega^I\}$ 构成的 SU(2) 三重态，其中满足 $I = 1, \ldots, 6$ ，此时左行世界片超流形式为

$$T_F(z) = \mathrm{i}\psi^\mu \partial X^\mu(z) + \mathrm{i}\sum_{I=1}^{6} \chi^I y^I \omega^I(z), \tag{30}$$

where the first term is due to the non-compact super-coordinates carrying the four-dimensional Lorentz indices, while the second term is due to the system of 18 free fermions. It is often convenient to intuitively think of y^I and ω^I as the auxiliary fermions arising from the fermionization of six internal bosonic coordinates $\mathrm{i}\partial X^I = y^I\omega^I$, compactified on circles of radius $R = \sqrt{\alpha'/2}$. In this sense, χ^I then plays the role of their fermionic superpartner.

> 其中第一项来自携带四维洛伦兹指标的非紧致超坐标，第二项则来自 18 个自由费米子构成的系统。通常很方便可以直观地将 y^I 和 ω^I 看作六个内部玻色坐标 $\mathrm{i}\partial X^I = y^I\omega^I$ 费米化后产生的辅助费米子，这些玻色坐标紧致化在半径为 $R = \sqrt{\alpha'/2}$ 的圆上。在此意义下，χ^I 就充当它们的费米超对称伴侣。

As we have mentioned, breaking the non-abelian local symmetry down to U(1)s can be accomplished by assigning different boundary conditions to the fermions χ^I, y^I, ω^I, realizing each of the SU(2)s. However, this assignment is highly constrained by the form (30) of the full worldsheet supercurrent, which must have well-defined periodicities in order for the $\mathcal{N} = 1$ superconformal theory to remain intact. Indeed, the first term in (30) is proportional to the worldsheet fermions ψ^μ transforming under the Lorentz group. If we denote their boundary conditions as $\begin{bmatrix} a \\ b \end{bmatrix}$, then the internal part $\chi^I y^I \omega^I$ should also carry the same net (anti)periodicities. [6]

> 如我们之前所述，将非阿贝尔局域对称性破缺到多个 U(1) 可以通过给实现各个 SU(2) 的费米子 χ^I, y^I, ω^I 分配不同边界条件来实现。但这种分配受到完整世界片超流形式 (30) 的严格限制，为了让 $\mathcal{N} = 1$ 超共形理论保持完好，超流必须有定义良好的周期性。确实，式 (30) 中的第一项正比于在洛伦兹群下变换的世界片费米子 ψ^μ。若我们将它们的边界条件记为 $\begin{bmatrix} a \\ b \end{bmatrix}$，那么内部部分 $\chi^I y^I \omega^I$ 也应当携带相同的净 (反) 周期性。[6]

Constraints from Modular Invariance

模不变性的约束

We have already mentioned that not all spin-structure assignments for the world-sheet fermions are allowed on topologically nontrivial worldsheets. Indeed, the vacuum amplitude of the theory at one and higher genera should remain invariant under large diffeomorphisms. Each such assignment of spin-structures consistent with (multi-loop) modular invariance gives rise to an a priori different string vacuum. These constraints first considered in [118, 119] were solved in full generality in [4] and [3], treating both the cases of real and complex fermions, and will be presented in section "Model Construction Rules, Spectrum, and Effective Superpotential".

我们已经提到，并非世界面费米子的所有自旋结构分配在拓扑非平凡世界面上都是被允许的。事实上，该理论在 1 亏格及更高亏格下的真空振幅应当在大微分同胚下保持不变。每个满足 (多圈) 模不变性的自旋结构分配都会先验地对应一个不同的弦真空。这些约束最早在文献 [118, 119] 中被研究，最终在文献 [4] 和 [3] 中得到了全通解，其中同时处理了实费米子和复费米子情形，我们将在“模型构造规则、谱和有效超势”一节展开介绍。

In this section, we discuss the conditions imposed by modular invariance in a simple heterotic setup where all fermions can be complexified. Although this is not the most general case, it is sufficient for outlining the salient features of the construction. The main idea is to impose modular invariance of the vacuum amplitude at one and higher genera, together with factorization, in order to extract conditions on the spin-structure-dependent coefficients.

本节我们讨论简单杂化 setup 下模不变性施加的条件，该 setup 中所有费米子都可以复化。尽管这不是最一般的情形，但足以说明该构造的核心特征。其核心思想是对 1 亏格及更高亏格的真空振幅施加模不变性，再结合分解性，从而推导出对依赖自旋结构的系数的约束条件。

At genus 1, the string worldsheet has the topology of a torus with Teichmüller parameter $\tau = \tau_1 + \mathrm{i}\tau_2$ to be integrated over the fundamental domain $\mathcal{F} = \mathcal{H}/\mathrm{SL}(2;\mathbb{Z})$ with $\mathcal{H}$ being the upper half-plane. Omitting an overall normalization constant, which is irrelevant for our discussion, the amplitude has the generic form

在 1 亏格下, 弦世界面具有环面拓扑, 其泰希米勒参数为 $\tau = \tau_1 + \mathrm{i}\tau_2$, 需要在基本域 $\mathcal{F} = \mathcal{H}/\mathrm{SL}(2;\mathbb{Z})$ 上积分，其中 $\mathcal{H}$ 是上半平面。省略对我们讨论无关的整体归一化常数后，振幅的一般形式为

$$F_1 = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2} \sum_{\substack{\text{spin}\\\text{structures}}} N\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix} Z_{b,c} Z_{\text{bos}\,c\beta,\gamma}\begin{bmatrix}a\\b\end{bmatrix} Z_{\text{long}}\begin{bmatrix}a\\b\end{bmatrix} Z\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix}(\tau,\bar{\tau}). \tag{31}$$

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[6] Strictly speaking, the supercurrent $T_F(z)$ in (30) should also include the contributions of the (super)ghost systems. We do not explicitly display these contributions here, since they will not play any particular role in our analysis. Alternatively, we may work in the light-cone and restrict ψ^μ and X^μ to the transverse directions only.

[6] 严格来说，(30) 式中的超流 $T_F(z)$ 还应当包含 (超) 鬼系统的贡献。由于这些贡献在我们的分析中不起特殊作用，我们在此不明确写出。或者，我们也可以采用光锥规范，将 ψ^μ 和 X^μ 仅限制在横向上。

We now describe the various contributions entering the above expression. $Z_{b,c}(\tau,\bar{\tau}) = \eta^2\bar{\eta}^2$ is the spin-structure-independent contribution of the $b, c, \bar{b}, \bar{c}$ ghosts, while $Z_{\text{bos}}(\tau,\bar{\tau}) = 1/\left(\sqrt{\tau_2}\eta\bar{\eta}\right)^4$ is the contribution of the non-compact worldsheet coordinates. As expected, the ghost system cancels the oscillator contributions of the longitudinal worldsheet coordinates X^0, X^1 . Furthermore, $Z_{\beta,\gamma}(\tau)$ is the contribution of the superghost system which, aside from a possible phase, exactly cancels against that of the longitudinal fermions ψ^0, ψ^1

现在我们来说明进入上述表达式的各项贡献。$Z_{b,c}(\tau,\bar{\tau}) = \eta^2\bar{\eta}^2$ 是 $b, c, \bar{b}, \bar{c}$ 鬼不依赖自旋结构的贡献，而 $Z_{\text{bos}}(\tau,\bar{\tau}) = 1/(\sqrt{\tau_2}\eta\bar{\eta})^4$ 是非紧致世界面坐标的贡献。不出所料，鬼系统抵消了纵向世界面坐标 X^0, X^1 的振荡贡献。此外，$Z_{\beta,\gamma}(\tau)$ 是超鬼系统的贡献，除了一个可能的相位外，它正好和纵向费米子 ψ^0, ψ^1 的贡献抵消

$$Z_{\text{long}}(\tau) = \frac{\vartheta\begin{bmatrix} a \\ b \end{bmatrix}}{\eta}, \tag{32}$$

since T_F, ψ^μ and the superghosts are required to share the same boundary conditions $\begin{bmatrix} a \\ b \end{bmatrix}$, in order to preserve the unbroken $\mathcal{N} = 1$ worldsheet supersymmetry. The block

这是因为要求 T_F, ψ^μ 和超鬼具有相同的边界条件 $\begin{bmatrix} a \\ b \end{bmatrix}$，才能保持未破缺的 $\mathcal{N} = 1$ 世界面超对称性。这个分块

$$Z\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}(\tau,\bar{\tau}) = \frac{\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(0;\tau)}{\eta(\tau)} \prod_A \frac{\vartheta\begin{bmatrix} a_A \\ b_A \end{bmatrix}(0;\tau)}{\eta(\tau)} \prod_{\bar{A}} \frac{\bar{\vartheta}\begin{bmatrix} a_{\bar{A}} \\ b_{\bar{A}} \end{bmatrix}(0;\bar{\tau})}{\bar{\eta}(\bar{\tau})}. \tag{33}$$

contains the contributions of the remaining left- and right-moving worldsheet fermions. In particular, the first factor carrying spin-structures $\begin{bmatrix} a \\ b \end{bmatrix}$ is the contribution of the transverse space-time directions ψ^3, ψ^4 . Note that we use here the symbol $\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}$ with $\mathbf{a} = (a, a_1, a_2, \ldots)$ and $\mathbf{b} = (b, b_1, b_2, \ldots)$ to collectively denote the spin-structure assignment of all left- and right-moving (complexified) fermions, while the indices A and $\bar{A}$ in .Eq. (33) run over the left- and right-moving complex fermions, respectively, except ψ^μ . Finally, the coefficients $N\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}$ are independent of the complex structure τ but depend on the spin-structure assignments and effectively correspond to choices of GGSO projections.

包含剩余左行和右行世界面费米子的贡献。具体而言，第一个承载自旋结构的因子 $\begin{bmatrix} a \\ b \end{bmatrix}$ 是横向时空方向 ψ^3, ψ^4 的贡献。请注意, 我们在此处使用符号 $\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}$ 搭配 $\mathbf{a} = (a, a_1, a_2, \ldots)$ 和 $\mathbf{b} = (b, b_1, b_2, \ldots)$ 来共同表示所有左行和右行 (复化) 费米子的自旋结构分配，而式 (33) 中的指标 A 和 $\bar{A}$ 分别遍历除 ψ^μ 之外的左行和右行复费米子。最后，系数 $N\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}$ 与复结构 τ 无关，但依赖于自旋结构分配，本质上对应 GGSO 投影的不同选择。

We are now ready to extract the constraints of one-loop modular invariance on the coefficients. To simplify the analysis and to set the constraints in their standard form in the literature, it is convenient to first switch to a slightly different convention for the Jacobi theta functions

我们现在可以导出单圈模不变性对系数的约束了。为简化分析，并将约束整理为文献中的标准形式，我们方便起见先对雅可比 θ 函数切换到一套略有不同的约定

$$\Theta\begin{bmatrix} a \\ b \end{bmatrix}(z;\tau) = \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}ab}\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z;\tau)\,, \tag{34}$$

which has the advantage of simpler periodicity properties in the lower argument:

这套约定的优势在于更低层自变量的周期性性质更简单:

$$\Theta\begin{bmatrix} a \\ b+2 \end{bmatrix}(z;\tau) = \Theta\begin{bmatrix} a \\ b \end{bmatrix}(z;\tau)\,,\ \Theta\begin{bmatrix} a+2 \\ b \end{bmatrix}(z;\tau) = \mathrm{e}^{-\mathrm{i}\pi b}\Theta\begin{bmatrix} a \\ b \end{bmatrix}(z;\tau)\,. \tag{35}$$

The one-loop vacuum amplitude then reads

单圈真空振幅可写为

$$F_1 = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2}\frac{1}{\tau_2\eta^2\overline{\eta}^2}\frac{1}{|\Xi|}\sum_{\mathbf{a},\mathbf{b}} C\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}\hat{Z}\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \tag{36}$$

where we used (34) to convert all ϑ ’ s into Θ ’ s and absorbed phase factors into the new constant coefficients

其中我们利用 (34) 将所有 ϑ 转换为 Θ ，并将相因子吸收到新的常数系数中

$$C\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}\mathbf{a}\cdot\mathbf{b}}N\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \tag{37}$$

where the dot product in the exponent is defined [7] in the Lorentzian sense, using the signature (10, 22) of the fermion charge lattice. Here, $\hat{Z}$ is simply given by (33) with ϑ ’ s replaced by Θ ’ s, and we also divide by the order $|\Xi|$ of the finite additive group [8] Ξ of boundary condition vectors a. Performing the transformation $\tau \to \tau + 1$ in (36), using the modular transformation properties of the Jacobi theta functions, resetting the summation as $\mathbf{b} \to \mathbf{b} - \mathbf{a} + \mathbf{1}$, and comparing with the original expression, it is easy to extract the first modular condition:

其中指数中的点积按洛伦兹意义定义 [7] ，采用费米子电荷格点的符号差 (10, 22) 。此处 $\hat{Z}$ 可直接由 (33) 将 ϑ 替换为 Θ 得到，我们还将结果除以边界条件向量 a 的有限加法群 [8] Ξ 的阶 $|\Xi|$ 。对 (36) 执行变换 $\tau \to \tau + 1$ ，利用雅可比 θ 函数的模变换性质，将求和重整理为 $\mathbf{b} \to \mathbf{b} - \mathbf{a} + \mathbf{1}$ 后与原式对比，很容易得到第一个模条件:

$$C\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \mathrm{e}^{\frac{\mathrm{i}\pi}{4}(\mathbf{a}\cdot\mathbf{a}+\mathbb{1}\cdot\mathbb{1})}C\begin{bmatrix} \mathbf{a} \\ \mathbf{b}-\mathbf{a}+\mathbf{1} \end{bmatrix}. \tag{38}$$

We also denote by1the boundary condition vector corresponding to periodic boundary conditions for all fermions. Note that the factor $\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\mathbb{1}\cdot\mathbb{1}} = -1$ in the heterotic string and reflects the transformation of the

Dedekind functions. [9] Similarly, requiring the invariance of (36) under the second modular transformation $\tau \to -1/\tau$ yields the second condition:

我们也将所有费米子都取周期性边界条件对应的边界条件向量记为 1。请注意，heterotic 弦中的因子 $e^{\frac{i\pi}{4}\mathbb{1}\cdot\mathbb{1}} = -1$ 反映了戴德金函数的变换。[9] 类似地，要求 (36) 在第二个模变换 $\tau \to -1/\tau$ 下不变，即可得到第二个条件:

$$C\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = e^{\frac{i\pi}{2}\mathbf{a}\cdot\mathbf{b}} C\begin{bmatrix} \mathbf{b} \\ -\mathbf{a} \end{bmatrix}. \tag{39}$$

Cluster decomposition requires the factorization of higher genus amplitudes when the donuts making up the Riemann surface are pulled infinitely far apart. This implies that the spin-structure coefficients at genus g must also factorize into products of one-loop coefficients:

簇分解要求，当构成黎曼曲面的环面被拉至无穷远时，高亏格振幅可以因子分解。这意味着亏格 g 下的自旋结构系数也必须可以分解为单圈系数的乘积:

$$C\begin{bmatrix} \mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \dots, \mathbf{a}_{(g)} \\ \mathbf{b}_{(1)}, \mathbf{b}_{(2)}, \dots, \mathbf{b}_{(g)} \end{bmatrix} = C\begin{bmatrix} \mathbf{a}_{(1)} \\ \mathbf{b}_{(1)} \end{bmatrix} C\begin{bmatrix} \mathbf{a}_{(2)} \\ \mathbf{b}_{(2)} \end{bmatrix} \dots C\begin{bmatrix} \mathbf{a}_{(g)} \\ \mathbf{b}_{(g)} \end{bmatrix}. \tag{40}$$

As a result, on a higher genus Riemann surface, Dehn twists acting on each separate donut alone leave the amplitude invariant as a consequence of one-loop modular invariance and factorization. In addition, factorization also ensures that Dehn twists mixing the spin structures of nearby donuts in the chain can be accounted for by imposing invariance of the two-loop vacuum amplitude under the nontrivial twist

因此，在高亏格黎曼曲面上，得益于单圈模不变性和因子分解性，作用在每个独立环面上的德恩扭转都会保持振幅不变。此外，因子分解性还保证，对于链中相邻环面混合自旋结构的德恩扭转，只需要要求两圈真空振幅在非平凡扭转下不变，就可以满足要求。

$$\Omega \to \Omega' = \Omega - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{41}$$

[7] For simplicity, we assumed all fermions are complexified so that only integer powers of thetas appear. It is easy to incorporate the case of half-integer powers, by scaling the corresponding elements of the dot product metric by 1/2 .

[7] 为简化起见，我们假设所有费米子都已复化，因此 θ 函数仅出现整数次幂。我们很容易推广到半整数次幂的情况，只需将点积度规的对应元素按 1/2 缩放即可。

[8] For (anti)periodic boundary conditions, the group Ξ is simply a direct sum of $\mathbb{Z}_2$ factors.

[8] 对于 (反) 周期性边界条件，群 Ξ 就是 $\mathbb{Z}_2$ 因子的直和。

[9] In type II theories, one instead has $e^{\frac{i\pi}{4}\mathbb{1}\cdot\mathbb{1}} = +1$, since the left- and right-moving worldsheets are both super-reparametrization invariant and the Dedekind functions arise in pairs, $\eta^{-12}\overline{\eta}^{-12}$.

[9] 在 II 型理论中，情况有所不同，存在 $e^{\frac{i\pi}{4}\mathbb{1}\cdot\mathbb{1}} = +1$，这是因为左行和右行世界面都具有超重参数化不变性，且戴德金函数成对出现，即 $\eta^{-12}\bar{\eta}^{-12}$。

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where Ω is the period matrix of the double torus. The generalization of (34) to genus 2 reads

其中 Ω 是双环面的周期矩阵。将式 (34) 推广到亏格 2 的情形可得

$$\Theta\left[\begin{array}{c}\alpha_{(1)},\alpha_{(2)}\\ \beta_{(1)},\beta_{(2)}\end{array}\right](z;\Omega) = \sum_{\mathbf{n}\in\mathbb{Z}^2} e^{i\pi\left(n-\frac{\alpha}{2}\right)^T\Omega\left(n-\frac{\alpha}{2}\right)+2\pi i\left(z-\frac{\beta}{2}\right)^T\left(n-\frac{\alpha}{2}\right)-\frac{i\pi}{2}\alpha^T\beta}, \tag{42}$$

where $\alpha = \left(\alpha_{(1)},\alpha_{(2)}\right)^T, \beta = \left(\beta_{(1)},\beta_{(2)}\right)^T$ are column vectors carrying the boundary conditions ascribed to the transport properties of fermions along the non-contractible cycles of each of the two donuts, and, similarly, z is a column vector of Jacobi parameters. It is then straightforward to obtain the transformation of the genus 2 theta constant ($z = 0$) under (41)

其中 $\alpha = \left(\alpha_{(1)},\alpha_{(2)}\right)^T, \beta = \left(\beta_{(1)},\beta_{(2)}\right)^T$ 是承载费米子沿两个环面各自不可收缩圈输运性质对应的边界条件的列向量，类似地，z 是雅可比参数构成的列向量。由此可以很容易得到亏格 2theta 常数 ($z = 0$) 在 (41) 下的变换

$$\Theta\left[\begin{array}{c}\alpha_{(1)},\alpha_{(2)}\\ \beta_{(1)},\beta_{(2)}\end{array}\right](0;\Omega') = e^{-\frac{i\pi}{2}\alpha_{(1)}\alpha_{(2)}}\Theta\left[\begin{array}{c}\alpha_{(1)},\alpha_{(2)}\\ \beta_{(1)}-\alpha_{(2)},\beta_{(2)}-\alpha_{(1)}\end{array}\right](0;\Omega).$$

(43)

Inserting this transformation into the genus 2 generalization of the vacuum amplitude (36) and imposing modular invariance, one finds the third and final condition

将该变换代入真空振幅 (36) 的亏格 2 推广形式，并要求模不变性，即可得到第三个也是最后一个条件

$$C\left[\begin{array}{c}\mathbf{a}\\ \mathbf{b}\end{array}\right]C\left[\begin{array}{c}\mathbf{a}'\\ \mathbf{b}'\end{array}\right] = \delta_a\delta_{a'}e^{-\frac{i\pi}{2}\mathbf{a}\cdot\mathbf{a}'}C\left[\begin{array}{c}\mathbf{a}\\ \mathbf{b}+\mathbf{a}'\end{array}\right]C\left[\begin{array}{c}\mathbf{a}'\\ \mathbf{b}'+\mathbf{a}\end{array}\right], \tag{44}$$

where the phases $\delta_a = (-1)^a, \delta_{a'} = (-1)^{a'}$ arise from the modular transformation properties of the two-loop worldsheet gravitino determinant, which can be uniquely fixed by requiring that the consistent 10d superstrings satisfy this condition. [10]

其中相位 $\delta_a = (-1)^a, \delta_{a'} = (-1)^{a'}$ 来自两圈世界面引力微子行列式的模变换性质，通过要求自洽的 10 维超弦满足该条件可以唯一确定这些相位。[10]

The three conditions (38), (39), and (44) together impose multi-loop modular invariance and can be solved in general [3,5], ensuring the absence of local anomalies, the correct spin-statistics connection, and unitarity. The solution corresponding to rational CFTs relevant for model building will be outlined in the next section. Before closing the discussion, however, it is useful to recast the two-loop condition (44) to a reduced form that severely constrains the phase dependence on the a-cycle and b-cycle boundary condition assignments. To this end, setting $\mathbf{a}' = \mathbf{c}'$ and $\mathbf{b}' = -\mathbf{a}$ in (44), and using (39) to bring $\mathbf{a}$ back to the upper characteristic, one finds

三个条件 (38)、(39) 和 (44) 共同给出了多圈模不变性约束，且一般情况下可解 [3,5]，保证了无局域反常、正确的自旋统计关系和幺正性。下一节将概述对应模型构建中相关有理共形场论的解。但在结束本节讨论前，我们不妨将两圈条件 (44) 改写为约化形式，该形式会对相位对 a-cycle 和 b-cycle 边界条件赋值的依赖产生极强约束。为此，在 (44) 中设 $\mathbf{a}' = \mathbf{c}'$ 和 $\mathbf{b}' = -\mathbf{a}$，再利用 (39) 将 $\mathbf{a}$ 还原为上特征标，可得

$$C\begin{bmatrix}\mathbf{a}\\ \mathbf{b}+\mathbf{c}\end{bmatrix} C\begin{bmatrix}\mathbf{c}\\ \mathbf{0}\end{bmatrix} = \delta_a\delta_c C\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} C\begin{bmatrix}\mathbf{a}\\ \mathbf{c}\end{bmatrix}. \tag{45}$$

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[10] In type II theories, δ_a should be identified with the space-time fermion parity of the theory, $(-1)^{F_L+F_R}$, receiving contributions from both the left- and the right-movers.

[10] 在 II 型理论中，δ_a 对应理论的时空费米子宇称，即 $(-1)^{F_L+F_R}$，同时接收左行模式和右行模式的贡献。

Clearly, $C\begin{bmatrix}\mathbf{c}\\ \mathbf{0}\end{bmatrix} \neq 0$ for all $\mathbf{c}$; otherwise, all coefficients would trivially vanish. Plugging $\mathbf{b} = \mathbf{c} = \mathbf{0}$ into the above equation and simplifying yields $C\begin{bmatrix}\mathbf{a}\\ \mathbf{0}\end{bmatrix} = \delta_a$, where we have conventionally set the overall normalization to $C\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix} = 1$. Substituting this into (45), we can then finally write the factorization condition as

显然，对所有 $\mathbf{c}$ 都有 $C\begin{bmatrix}\mathbf{c}\\ \mathbf{0}\end{bmatrix} \neq 0$；否则所有系数都会平凡地等于零。将 $\mathbf{b} = \mathbf{c} = \mathbf{0}$ 代入上述方程化简后得到 $C\begin{bmatrix}\mathbf{a}\\ \mathbf{0}\end{bmatrix} = \delta_a$，此处我们按惯例将整体归一化设为 $C\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix} = 1$。将其代入 (45)，我们最终可以将因子化条件写为

$$C\begin{bmatrix}\mathbf{a}\\ \mathbf{b}+\mathbf{c}\end{bmatrix} = \delta_a C\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} C\begin{bmatrix}\mathbf{a}\\ \mathbf{c}\end{bmatrix}. \tag{46}$$

This form will be particularly useful when we discuss the connection of the FFF to toroidal orbifolds in section "Map to Orbifolds".

当我们在“映射到 orbifold”一节讨论 FFF 与环面 orbifold 的联系时，这个形式会格外有用。

Model Construction Rules, Spectrum, and Effective Superpotential

模型构造规则、能谱与有效超势

The modular invariance constraints discussed in section ”Constraints from Modular Invariance” can be solved in terms of a set of N basis vectors $B = \{\beta_1, \ldots, \beta_N\}$ which encode the boundary conditions of the worldsheet fermions and a set of phases $c\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix}, i, j = 1, \ldots N$ associated with generalized Gliozzi-Scherk-Olive (GGSO) projections, called spin-structure coefficients.

在“模不变性的约束”章节讨论的模不变性约束可以通过一组 N 基矢 $B = \{\beta_1, \ldots, \beta_N\}$ 和一组相位 $c\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix}, i, j = 1, \ldots N$ 求解，其中基矢 $B = \{\beta_1, \ldots, \beta_N\}$ 编码了世界面费米子的边界条件，相位 $c\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix}, i, j = 1, \ldots N$ 与广义格里奥齐-舍克-奥利维 (GGSO) 投影相关，被称为自旋结构系数。

In four space-time dimensions, the standard notation of worldsheet fermionic fields used in the FFF model building is as follows. The left-moving fermions comprise 20 real fields. These are ψ^μ that stands for the two space-time fermions in the light-cone gauge, the real fermions $\chi^1, \ldots, \chi^6$ that parameterize the six fermionic internal coordinates, and the 12 real fermions $y^1, \omega^1, \ldots, y^6, \omega^6$ that come from the fermionization of the associated internal bosonic coordinates. The right-moving fields consist of 12 real fermions $\bar{y}^1, \overline{\omega}^1, \ldots, \bar{y}^6, \overline{\omega}^6$ ascribed to the fermionization of the internal bosonic coordinates and 16 complex fermions denoted as $\overline{\psi}^1, \ldots, \overline{\psi}^5, \overline{\eta}^1, \overline{\eta}^2, \overline{\eta}^3, \overline{\phi}^1, \ldots, \overline{\phi}^8$ for reasons that will become apparent later. As explained in section ”Constraints from Modular Invariance”, worldsheet supersymmetry is nonlinearly realized among the left-moving fermions χ^I, y^I, ω^I . Although there are several such realizations [3], we focus on the simplest case exploited in heterotic model building where the supercurrent takes the form (30). In this description, each basis vector consists of a set of phases, e.g.,

在四维时空中, FFF 模型构建中采用的世界面费米场标准记号如下。左行费米子包含 20 个实场: 它们是代表光锥规范中两个时空费米子的 ψ^μ 、参数化六个内部费米坐标的实费米子 $\chi^1, \ldots, \chi^6$, 以及从关联内部玻色坐标费米化得到的 12 个实费米子 $y^1, \omega^1, \ldots, y^6, \omega^6$ 。右行场包含 12 个来自内部玻色坐标费米化的实费米子 $\bar{y}^1, \overline{\omega}^1, \ldots, \bar{y}^6, \overline{\omega}^6$, 以及 16 个复费米子, 我们将其记作 $\overline{\psi}^1, \ldots, \overline{\psi}^5, \overline{\eta}^1, \overline{\eta}^2, \overline{\eta}^3, \overline{\phi}^1, \ldots, \overline{\phi}^8$, 后续会说明这样做的原因。正如“模不变性的约束”章节所述, 世界面超对称在左行费米子 χ^I, y^I, ω^I 之间非线性实现。尽管这类实现已有多种 [3]，我们聚焦于杂化弦模型构建中采用的最简情形，该情形下超流取形式 (30)。在这套描述中，每个基矢由一组相位构成，例如:

$$\beta = \{\alpha(\psi^\mu), \alpha(\chi^{12}), \alpha(\chi^{34}), \alpha(\chi^{56}), \alpha(y^1), \ldots, \alpha(y^6), \alpha(\omega^1), \ldots, \alpha(\omega^6);$$

$$\alpha(\bar{y}^1), \ldots, \alpha(\bar{y}^6), \alpha(\overline{\omega}^1), \ldots, \alpha(\overline{\omega}^6),$$

$$\alpha(\overline{\psi}^1), \ldots, \alpha(\overline{\psi}^5), \alpha(\overline{\eta}^1), \alpha(\overline{\eta}^2), \alpha(\overline{\eta}^3), \alpha(\overline{\phi}^1), \ldots, \alpha(\overline{\phi}^8)\},$$

(47)

portraying the parallel transport properties of the worldsheet fermions

描述世界面费米子的平行输运性质

$$f \to -\mathrm{e}^{\mathrm{i}\pi\alpha(f)} f, \tag{48}$$

where $\alpha(f) \in (-1,1]$ are in general fractional numbers, with the special cases $\alpha = 0, 1$ corresponding to antiperiodic (NS) and periodic (R) fermions, respectively. The semicolon in (47) separates left- and right-moving fermions.

其中 $\alpha(f) \in (-1,1]$ 一般为分数，特例 $\alpha = 0, 1$ 分别对应反周期 (NS) 费米子和周期 (R) 费米子。(47) 中的分号分隔了左行费米子和右行费米子。

The partition function of the theory can be expressed as a sum over pairs of spin structures in an abelian group Ξ spanned by the basis vectors, $\Xi = \{\xi \mid \xi = \ m_1\beta_1 + \cdots + m_n\beta_N, m_i = 0, \ldots, N_i\}$, with N_i the smallest integer for which $N_i\beta_i = 0 \mod 2$,

该理论的配分函数可以表示为对基矢张成的阿贝尔群 Ξ 中自旋结构对的求和，即 $\Xi = \{\xi \mid \xi = m_1\beta_1 + \cdots + m_n\beta_N, m_i = 0, \ldots, N_i\}$，其中 N_i 是满足 $N_i\beta_i = 0 \mod 2$ 的最小整数，

$$Z = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^3 \eta^{12} \overline{\eta}^{24}} \frac{1}{2^N} \sum_{\alpha,\beta\in\Xi} c\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \prod_{f\in \text{real left}} \Theta^{\frac{1}{2}}\begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix} \prod_{f\in \text{complex left}} \Theta\begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix}$$
$$\prod_{f\in \text{real right}} \overline{\Theta}^{\frac{1}{2}}\begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix} \prod_{f\in \text{complex right}} \overline{\Theta}\begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix}.$$

(49)

In the last expression, the products extend over real/complex left/right fermions, respectively. Here $\Theta\begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix}$ is the Jacobi theta function with characteristics, η stands for the Dedekind eta function, and $\mathcal{F}$ is the fundamental domain.

在上一表达式中，乘积分别对实/复左/右费米子遍历。此处 $\Theta\begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix}$ 是带特征的雅可比 θ 函数，η 代表戴德金 η 函数，$\mathcal{F}$ 是基本域。

Basis vectors and spin-structure coefficients are subject to constraints imposed by modular invariance and factorization of the string amplitudes (38), (39), (44). The basis vectors must satisfy

基矢和自旋结构系数满足模不变性和弦振幅因子化 (38)、(39)、(44) 施加的约束。基矢必须满足

$$N_{ij}\beta_i \cdot \beta_j = 0 \mod 4 \text{ , with } N_{ij} = \mathrm{lcm}\left(N_i, N_j\right) \text{ ,} \tag{50}$$

$$N_i\beta_i \cdot \beta_i = 0 \mod 8 \text{ , if } N_i \text{ is even,}$$

along with the requirement that the number of real fermions which are periodic in any combination of four basis vectors has to be even. Moreover, to guarantee well-defined periodicity properties of the supercurrent (30), we have to impose

同时要求任意四个基矢的任意组合中，周期实费米子的数量必须为偶数。此外，为保证超流 (30) 具有定义良好的周期性，我们必须要求

$$\beta_i\left(\chi^I\right)+\beta_i\left(y^I\right)+\beta_i\left(\omega^I\right)=\beta_i\left(\psi^\mu\right) \mod 2 \forall I=1,\ldots,6. \tag{51}$$

The spin-structure coefficients $c\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix}$ can be expressed in terms of $N(N-1)/2+1$ independent phases, namely, $c\begin{bmatrix}1\\ 1\end{bmatrix}, c\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix}, i>j=1,\ldots,N$, where $\mathbb{1}\in\Xi$ is the vector with all fermions periodic. Furthermore, composite spin-structure coefficients, as the ones appearing in (49), can be reduced utilizing

自旋结构系数 $c\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix}$ 可以用 $N(N-1)/2+1$ 个独立相位表示, 即 $c\begin{bmatrix}1\\ 1\end{bmatrix}, c\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix}, i>j=1,\ldots,N$ ，其中 $\mathbb{1}\in\Xi$ 是所有费米子都为周期性的矢量。此外，复合自旋结构系数，如出现在式 (49) 中的那些，可以通过利用

$$\begin{aligned}
c\begin{bmatrix}\alpha\\ \beta+\gamma\end{bmatrix}&=\delta_\alpha c\begin{bmatrix}\alpha\\ \beta\end{bmatrix}c\begin{bmatrix}\alpha\\ \gamma\end{bmatrix},\\
c\begin{bmatrix}\alpha\\ \beta\end{bmatrix}&=\mathrm{e}^{\mathrm{i}\pi(\alpha\cdot\beta)/2}c\begin{bmatrix}\beta\\ \alpha\end{bmatrix}^*,\\
c\begin{bmatrix}\alpha\\ \alpha\end{bmatrix}&=\mathrm{e}^{\mathrm{i}\pi(\alpha\cdot\alpha+\mathbb{1}\cdot\mathbb{1})/4}c\begin{bmatrix}\alpha\\ \mathbb{1}\end{bmatrix},
\end{aligned} \tag{52}$$

where $\delta_\alpha=(-1)^{\alpha(\psi^\mu)}$, and the Lorentzian dot product is defined as follows:

其中 $\delta_\alpha=(-1)^{\alpha(\psi^\mu)}$ ，洛伦兹点积定义如下:

$$\alpha\cdot\beta=\left\{\frac{1}{2}\sum_{f\in\left\{\substack{\text{real}\\ \text{left}}\right\}}+\sum_{f\in\left\{\substack{\text{complex}\\ \text{left}}\right\}}-\frac{1}{2}\sum_{f\in\left\{\substack{\text{real}\\ \text{right}}\right\}}-\sum_{f\in\left\{\substack{\text{complex}\\ \text{right}}\right\}}\right\}\alpha(f)\beta(f). \tag{53}$$

The Hilbert space of states contributing to (49) can be recast in the form

对式 (49) 有贡献的态的希尔伯特空间可以改写为如下形式

$$\mathcal{H}=\bigoplus_{\alpha\in\Xi}\prod_{i=1}^{N}\left\{\mathrm{e}^{\mathrm{i}\pi\beta_i F_\alpha}=\delta_\alpha c\begin{bmatrix}\alpha\\ \beta_i\end{bmatrix}^*\right\}\mathcal{H}_\alpha, \tag{54}$$

where $\mathcal{H}_\alpha$ is the Hilbert space of the sector α , and the expression in curly brackets represents a GSO projection that projects onto states satisfying $\mathrm{e}^{i\pi\beta_i F_\alpha}=\delta_\alpha c\begin{bmatrix}\alpha\\ \beta_i\end{bmatrix}^*$. Here,

其中 $\mathcal{H}_{\alpha}$ 是扇区 α 的希尔伯特空间，大括号内的表达式表示 GSO 投影，将投影到满足 $e^{i\pi\beta_i F_\alpha} = \delta_\alpha c\begin{bmatrix}\alpha\\ \beta_i\end{bmatrix}^*$ 的态上。在此，

$$\beta_i F_\alpha = \left\{\sum_{f\in\alpha_L} - \sum_{f\in\alpha_R}\right\}\beta_i(f)F_\alpha(f) \tag{55}$$

where $\alpha = \{\alpha_\mathrm{L};\alpha_\mathrm{R}\}$ and $F_\alpha(f)$ stands for the fermion number operator which takes values +1 or -1 when acting on f or f^* , respectively. When the vacuum is degenerate, our convention implies $F(f) = 0$ and $F(f) = -1$ for a state annihilated by f_0 and f_0^* , respectively.

其中 $\alpha = \{\alpha_\mathrm{L};\alpha_\mathrm{R}\}$，$F_\alpha(f)$ 是费米子数算符: 分别作用在 f 和 f^* 上时，其本征值为 +1 或-1。当真空简并时，按我们的约定，被 f_0 和 f_0^* 湮灭的态分别满足 $F(f) = 0$ 和 $F(f) = -1$ 。

The mass formula for string states in a sector $\alpha = \{\alpha_\mathrm{L};\alpha_\mathrm{R}\}$ is

扇区 $\alpha = \{\alpha_\mathrm{L};\alpha_\mathrm{R}\}$ 中弦态的质量公式为

$$M_\alpha^2 = -\frac{1}{2} + \frac{1}{8}\alpha_\mathrm{L}\cdot\alpha_\mathrm{L} + N_\mathrm{L} = -1 + \frac{1}{8}\alpha_\mathrm{R}\cdot\alpha_\mathrm{R} + N_\mathrm{R},$$

where $N_\mathrm{L}, N_\mathrm{R}$ stand for (sums of) left/right oscillator frequencies, respectively. For a fermion transforming as in Eq. (48), the oscillator frequencies are $[(1+\alpha(f))/2+$ integer$]$ for f and $[(1-\alpha(f))/2+$ integer $]$ for f^* . Using the identity $\delta_0 c\begin{bmatrix}0\\ \beta_i\end{bmatrix}^* = \delta_{\beta_i}$, it can be easily shown that the massless spectrum always includes a state of the form $\psi^\mu_{\frac{1}{2}}\left(\bar\partial\bar X\right)^\mu_1|0\rangle$ which arises from the 0-sector (NS) and contains the graviton, | the dilaton, and the two-index antisymmetric tensor. Similarly, we can infer that the 0-sector spectrum does not depend on the GGSO coefficients, but only on the choice of the basis vectors. Moreover, it turns out that the presence of space-time supersymmetry (SUSY), and the absence of tachyons, is ensured by including the vector $S = \{\psi^\mu, \chi^1, \ldots, \chi^6\}$ in the defining basis set and choosing the relevant spin-structure coefficients such that the associated gravitino multiplet survives.

其中 $N_\mathrm{L}, N_\mathrm{R}$ 分别代表左/右振荡器频率 (之和)。对于如式 (48) 变换的费米子，振荡器频率对 f 为 $[(1+\alpha(f))/2+$整数$]$,对 f^* 为 $[(1-\alpha(f))/2+$ integer $]$。利用恒等式 $\delta_0 c\begin{bmatrix}0\\ \beta_i\end{bmatrix}^* = \delta_{\beta_i}$,可轻易证明无质量谱总是包含形式为 $\psi^\mu_{\frac{1}{2}}\left(\bar\partial\bar X\right)^\mu_1|0\rangle$ which arises from the 0-sector (NS) and contains the graviton, | 的态:dilaton(dilaton 即 dilaton, 此处保留物理名词), 以及双指标反对称张量。同理可推得 0 扇区谱不依赖于 GGSO 系数, 仅依赖于基矢的选择。此外可证明, 若在定义基集中包含矢量 $S = \{\psi^\mu, \chi^1, \ldots, \chi^6\}$ 并选择相关自旋结构系数使得伴随引力微子多重态保留，即可保证存在时空超对称性 (SUSY) 且不存在快子。

The low-energy effective theory of a generic heterotic string model in the FFF is a $\mathcal{N} = 1$ no-scale supergravity [38, 57, 58, 126] . The Kähler potential of chiral multiplets is exactly calculable at string tree-level at every order in α' [10, 70, 70, 71, 131] . Furthermore, the superpotential of the effective theory is also calculable order by order in the α' -expansion [114,115] at string tree-level and receives no string loop corrections [53, 149]. Superpotential calculation amounts to evaluating correlation functions of the associated primary fields which in the case of unpaired real fermions incorporate Ising fields [49] that result in nontrivial elimination of superpotential couplings normally allowed by gauge symmetries. For example, trilinear superpotential couplings reduce to correlators involving two or three Ising fields of which only the following are nonvanishing:

FFF 框架下一般杂化弦模型的低能有效理论是 $\mathcal{N} = 1$ 无标度超引力 [38, 57, 58, 126] 。手征多重态的凯勒势能可在弦树图阶对 α' [10, 70, 70, 71, 131] 任意阶精确计算。此外，有效理论的超势能也可在弦树图阶按 α' 展开逐阶计算 [114,115]，且不存在弦圈修正 [53, 149]。超势能计算等价于计算相关初级场的关联函数，对于未配对实费米子，该计算包含伊辛场 [49]，这些伊辛场会非平凡地消除规范对称性通常允许的超势能耦合。例如，三线性超势能耦合退化为包含两到三个伊辛场的关联函数，其中仅以下关联函数非零:

$$\langle \sigma_\pm \sigma_\pm \rangle = \langle f f \rangle = \langle \bar{f} \bar{f} \rangle = 1, \tag{56}$$

$$\langle \sigma_+ \sigma_- f \rangle = \langle \sigma_+ \sigma_- \bar{f} \rangle = 1/\sqrt{2},$$

where σ_+, σ_- and $f/\bar{f}$ refer to the order, disorder, and fermion operators, respectively. The contribution of the complex fields that pertain to the internal fermionic coordinates $\chi^1, \ldots, \chi^6$ associated with a conserved $\mathrm{U}(1)$ current of the $\mathcal{N} = 2$ worldsheet supersymmetry algebra leads to additional restrictive selection rules for the superpotential couplings [129, 143].

其中 σ_+, σ_- 和 $f/\bar{f}$ 分别对应有序、无序和费米子算符。属于内部费米坐标 $\chi^1, \ldots, \chi^6$ 的复场，其对 $\mathcal{N} = 2$ 世界面超对称代数的守恒 $\mathrm{U}(1)$ 流的贡献，会为超势能耦合带来额外的限制性选择定则 [129, 143]。

Let us close this session by presenting an illustrative model. We consider the following ten-element basis $B = \{\beta_1, \ldots, \beta_{10}\}$[11]

最后我们通过一个示例模型结束本节。我们考虑如下由十个元素构成的基 $B = \{\beta_1, \ldots, \beta_{10}\}$[11]

$$\beta_1 = \mathbb{1} = \left\{\psi^\mu, \chi^{1\ldots6}, y^{1\ldots6}, \omega^{1\ldots6}; \bar{y}^{1\ldots6}, \overline{\omega}^{1\ldots6}, \overline{\psi}^{1\ldots5}, \overline{\psi}^{6\ldots10}, \overline{\psi}^{11\ldots15}, \overline{\eta}\right\},$$

$$\beta_2 = S = \left\{\psi^\mu, \chi^{1\ldots6}\right\},$$

$$\beta_{2+i} = e_i = \left\{y^i, \omega^i; \bar{y}^i, \overline{\omega}^i\right\}, i = 1 \ldots 6,$$

$$\beta_9 = b_1 = \left\{x^{34}, \chi^{56}, y^{3456}; \bar{y}^{3456}, \overline{\psi}^{1\ldots5}, \overline{\eta}\right\},$$

$$\beta_{10} = b_2 = \left\{\chi^{12}, x^{56}, y^{1256}; \bar{y}^{1256}, \overline{\psi}^{6\ldots10}, \overline{\eta}\right\}.$$

(57)

This defines an $\mathcal{N} = 1$ supersymmetric [12] $\mathrm{SO}(10)^3 \times \mathrm{U}(1)$ model, where the three $\mathrm{SO}(10)$ factors are associated with $\overline{\psi}^{1\ldots5}$, $\overline{\psi}^{6\ldots10}$, and $\overline{\psi}^{11\ldots15}$, respectively, and the $\mathrm{U}(1)$ factor is related to $\overline{\eta}$. The untwisted sector $(S)+0^{13}$ matter spectrum is independent of the GGSO coefficient choice and can be fully derived using the basis (57). As seen from Table 1, it includes two vectorials from each SO(10) with opposite U(1) charges, three bi-vectorials and six total singlets. The untwisted sector spectrum depends on the choice of $c\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix}$. However, there is a systematic way to derive the full spectrum. Massless states transforming in the spinorial representations of the three SO(10) factors can arise from the sectors $\mathcal{S}^1_{p^1q^1r^1s^1} = (S+)\,b_1+p^1e_3+q^1e_4+r^1e_5+s^1e_6$, $\mathcal{S}^2_{p^2q^2r^2s^2} = (S+)\,b_2+p^2e_1+q^2e_2+r^2e_5+s^2e_6$ and $\mathcal{S}^3_{p^3q^3r^3s^3} = (S+)\,b_3+p^3e_1+q^3e_2+r^3e_3+s^3e_4$, where $p^I,q^I,r^I,s^I=0,1$ for $I=1,2,3$ and $b_3=b_1+b_2+x$ with $x=\mathbb{1}+S+\sum_{i=1}^{6}e_i$. Each of the sectors $\mathcal{S}^I_{\rm pqrs}$, $I=1,2,3$ can provide one spinorial, so in total we can have 16 spinorials for each SO(10) group. However, depending on the choice of the GGSO coefficients, the number of spinorials can be reduced. This is most easily understood by considering GGSO projections of nonoverlapping basis vectors. Note that $e_1\cap\mathcal{S}^1_{pqrs}=e_2\cap\mathcal{S}^1_{pqrs}=\varnothing$, and, similarly, $e_3\cap\mathcal{S}^2_{pqrs}=e_4\cap\mathcal{S}^2_{pqrs}=\varnothing$, $e_5\cap\mathcal{S}^3_{pqrs}=e_6\cap\mathcal{S}^3_{pqrs}=\varnothing$. Namely, the relative GGSO projections in the sectors $\mathcal{S}^I_{pqrs}$ are

这定义了一个 $\mathcal{N}=1$ 超对称 [12]$\mathrm{SO}(10)^3\times\mathrm{U}(1)$ 模型，其中三个 SO (10) 因子分别对应 $\overline{\psi}^{1\ldots5},\overline{\psi}^{6\ldots10}$ 和 $\overline{\psi}^{11\ldots15}$，U (1) 因子对应 $\overline{\eta}$。未扭区 $(S)+0^{13}$ 物质谱与 GGSO 系数选择无关，可利用基 (57) 完整推导。如表 1 所示，谱中包含每个 SO(10) 提供的两个带相反 U (1) 电荷的向量、三个双向量以及总共六个单态。未扭区谱依赖于 $c\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix}$ 的选择，但存在一套系统方法推导完整谱。属于三个 SO (10) 因子旋量表示的无质量态可以从区 $\mathcal{S}^1_{p^1q^1r^1s^1} = (S+)\,b_1+p^1e_3+q^1e_4+r^1e_5+s^1e_6$, $\mathcal{S}^2_{p^2q^2r^2s^2} = (S+)\,b_2+p^2e_1+q^2e_2+r^2e_5+s^2e_6$ 和 $\mathcal{S}^3_{p^3q^3r^3s^3} = (S+)\,b_3+p^3e_1+q^3e_2+r^3e_3+s^3e_4$ 产生，其中对于 $I=1,2,3$ 和 $b_3=b_1+b_2+x$ 满足 $p^I,q^I,r^I,s^I=0,1$，且满足 $x=\mathbb{1}+S+\sum_{i=1}^{6}e_i$。每个 $\mathcal{S}^I_{\rm pqrs}$, $I=1,2,3$ 区可以提供一个旋量，因此每个 SO(10) 群总共可以得到 16 个旋量。但旋量数目会随 GGSO 系数的选择减少，这一点通过分析不重叠基矢的 GGSO 投影最容易理解。注意到 $e_1\cap\mathcal{S}^1_{pqrs}=e_2\cap\mathcal{S}^1_{pqrs}=\varnothing$，同理可得 $e_3\cap\mathcal{S}^2_{pqrs}=e_4\cap\mathcal{S}^2_{pqrs}=\varnothing$、$e_5\cap\mathcal{S}^3_{pqrs}=e_6\cap\mathcal{S}^3_{pqrs}=\varnothing$。即，区 $\mathcal{S}^I_{pqrs}$ 中的相对 GGSO 投影为

$$c\begin{bmatrix}\mathcal{S}^I_{pqrs}\\ e_{2I-1}\end{bmatrix}^* = c\begin{bmatrix}\mathcal{S}^I_{pqrs}\\ e_{2I}\end{bmatrix}^* = -1. \tag{58}$$

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[11] Here we denote the 16 complex right-moving fermions as follows: $\overline{\psi}^{1\ldots5},\overline{\psi}^{6\ldots10},\overline{\psi}^{11\ldots15},\overline{\eta}$.

[11] 在此我们将 16 个复右行费米子标记如下: $\overline{\psi}^{1\ldots5},\overline{\psi}^{6\ldots10},\overline{\psi}^{11\ldots15},\overline{\eta}$。

[12] Provided we choose the GGSO coefficients $c\begin{bmatrix}S\\ e_i\end{bmatrix}=-1, i=1,\ldots,6$.

[12] 前提是我们选择 GGSO 系数 $c\begin{bmatrix}S\\ e_i\end{bmatrix}=-1, i=1,\ldots,6$。

[13] Hereafter, we use a compact notation for sectors contributing to the same field multiplet, for example, $(S)+0$ stands for sectors S and 0.

[13] 在下文中，我们对贡献到同一个场多重态的扇区使用紧凑标记，例如，$(S)+0$ 代表扇区 S 和 0。

Table 1 Massless matter states and $\mathrm{SO}(10)^3 \times \mathrm{U}(1)$ quantum numbers of the model defined in (57) and (60)

表 1 (57) 和 (60) 定义模型的无质量物质态与 $\mathrm{SO}(10)^3 \times \mathrm{U}(1)$ 量子数

Sector	Field	SO(10)	SO(10)	SO(10)	U(1)	Multiplicity
(S) +0	h_1	10	1	1	+1	1
	$\bar{h}_1$	10	1	1	-1	1
	h_2	1	10	1	+1	1
	$\bar{h}_2$	1	10	1	-1	1
	h_3	1	1	10	+1	1
	$\bar{h}_3$	1	1	10	-1	1
	H_{12}	10	10	1	0	1
	H_{13}	10	1	10	0	1
	H_{23}	1	10	10	0	1
	$\Phi_i, i=1,\ldots,6$	1	1	1	0	6
(S) + b	S_1	16	1	1	+1/2	1
(S) + b $t_1+e_5+e_6$	S_1'	16	1	1	-1/2	1
(S) + b $_1+e_3+e_4+e_5+e_6$	S_1''	16	1	1	+1/2	1
$(S)+b_1+e_3+e_4$	$\bar{S}_1$	16	1	1	+1/2	1
$(S)+b_2+e_5+e_6$	S_2	1	16	1	+1/2	1
$(S)+b_2+e_1+e_6$	S_2'	1	16	1	-1/2	1
$(S)+b_2+e_1+e_2+e_5+e_6$	S_2''	1	16	1	+1/2	1
$(S)+b_2+e_2+e_6$	$\bar{S}_2$	1	16	1	+1/2	1
$(S)+b_3+e_1+e_2$	S_3	1	1	16	+1/2	1
$(S)+b_3+e_1$	S_3'	1	1	16	+1/2	1
$(S)+b_3+e_4$	S_3''	1	1	16	+1/2	1
$(S)+b_3+e_2+e_4$	$\bar{S}_3$	1	1	16	-1/2	1

These projections can reduce the number of spinorials by a factor of 4, which leads to four spinorials for each SO(10). The spinorial chiralities can be readily obtained using appropriate GGSO projections. Notice that $\mathbb{1}+b_2+e_3+e_4+re_5+se_6 \cap \mathcal{S}^1_{\mathrm{pqrs}} = \left\{\psi^\mu, \overline{\psi}^{1\ldots5}\right\}$ and analogously $\mathbb{1}+b_1+e_1+e_2+re_5+se_6 \cap \mathcal{S}^2_{\mathrm{pqrs}} = \left\{\psi^\mu, \overline{\psi}^{6\ldots10}\right\}, \mathbb{1}+b_1+e_1+e_2+re_3+se_4 \cap \mathcal{S}^3_{pqrs} = \left\{\psi^\mu, \overline{\psi}^{11\ldots15}\right\}$. The associated GGSO projections yield the chirality χ^1_{pqrs} of the spinorials of the first SO (10) gauge symmetry factor

这些投影可以将旋量的数量缩减为原来的 1/4，使得每个 SO(10) 对应四个旋量。旋量手征可以通过合适的 GGSO 投影轻松得到。注意 $\mathbb{1}+b_2+e_3+e_4+re_5+se_6 \cap \mathcal{S}^1_{\mathrm{pqrs}} = \left\{\psi^\mu, \overline{\psi}^{1\ldots5}\right\}$，类似地 $\mathbb{1}+b_1+e_1+e_2+re_5+se_6 \cap \mathcal{S}^2_{\mathrm{pqrs}} = \left\{\psi^\mu, \overline{\psi}^{6\ldots10}\right\}, \mathbb{1}+b_1+e_1+e_2+re_3+se_4 \cap \mathcal{S}^3_{pqrs} = \left\{\psi^\mu, \overline{\psi}^{11\ldots15}\right\}$。相关的 GGSO 投影得出第一个 SO (10) 规范对称因子旋量的手征 χ^1_{pqrs}

$$\chi^1_{pqrs} = -\operatorname{ch}(\psi^\mu)\, c\begin{bmatrix} S^1_{pqrs} \\ \mathbb{1} + b_2 + e_3 + e_4 + re_5 + se_6 \end{bmatrix}^*, \quad (59)$$

where $\operatorname{ch}(\psi^\mu)$ is the space-time chirality and similarly for the chiralities $\chi^I_{pqrs}, I = 2{,}3$ of the other two SO(10) factors. More particularly for the choice

其中 $\operatorname{ch}(\psi^\mu)$ 是时空手征，另外两个 SO(10) 因子的手征 $\chi^I_{pqrs}, I = 2$、3 同理。对于该选择具体有

$$c\begin{bmatrix} \beta_i \\ \beta_j \end{bmatrix} = (-1)^{u_{ij}}, u = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad (60)$$

we obtain the matter spectrum of Table 1. Note that the U_1 factor is anomalous. As inferred from Table 1,

我们得到表 1 的物质谱。注意 U_1 因子是反常的。如表 1 所示,

$$\operatorname{Tr}(U_1) = 24 \quad (61)$$

The tree-level superpotential is

树-level 超势为

$$W_3 = H_{12}H_{13}H_{23} + H_{12}\left(h_1\bar{h}_2 + h_2\bar{h}_1\right) + H_{13}\left(h_1\bar{h}_3 + h_3\bar{h}_1\right) + H_{23}\left(h_2\bar{h}_3 + h_3\bar{h}_2\right)$$

$$+\bar{h}_1\left(S_1^2 + S_1''^2 + \bar{S}_1^2\right) + \bar{h}_2\left({S_3}^2 + S_3'^2 + S_3''^2\right) + \bar{h}_3\left({S_2}^2 + S_2''^2 + \bar{S}_2^2\right)$$

$$+h_1S_1'^2 + h_3S_2'^2 + h_2\bar{S}_3^2.$$

(62)

This model is very similar to the orbifold model constructed in [15] in the context of ”string GUTs” discussed also in [29].

该模型与文献 [15] 在“弦大统一理论”背景下构建的 orbifold 模型非常相似，文献 [29] 也讨论了该模型。

$\mathcal{N} = 1$ Supersymmetric Models

$\mathcal{N} = 1$ 超对称模型

The FFF has yielded several phenomenologically interesting $\mathcal{N} = 1$ super-symmetric models, with features close to the minimal supersymmetric standard model (MSSM). Among the first models built, and most studied, are the flipped $\mathrm{SU}(5) \times \mathrm{U}(1)$ model [11-13], the Pati-Salam model [19,132], and the Standard-like Model [92]. These are derived from an SO(10) embedding realized using a common set of seven basis vectors

FFF 框架得到了多个唯象学上有研究价值的 $\mathcal{N} = 1$ 超对称模型，其性质与最小超对称标准模型 (MSSM) 十分接近。最早构建且研究最多的模型包括翻转 $\mathrm{SU}(5) \times \mathrm{U}(1)$ 模型 [11-13]、帕蒂-萨拉姆模型 [19,132] 和类标准模型 [92]。这些模型均源自 SO(10) 嵌入，由一组共 7 个基矢实现

$$
\begin{aligned}
\beta_1 &= \zeta = \left\{\overline{\phi}^{1\ldots8}\right\},\\
\beta_2 &= S = \left\{\psi^\mu, \chi^{1\ldots6}\right\},\\
\beta_3 &= b_1 = \left\{\psi^\mu, \chi^{12}, y^{3\ldots6}; \bar{y}^{3\ldots6}, \overline{\psi}^{1\ldots5}, \overline{\eta}^{1}\right\},\\
\beta_4 &= b_2 = \left\{\psi^\mu, \chi^{34}, y^{12}, \omega^{56}; \bar{y}^{12}, \overline{\omega}^{56}, \overline{\psi}^{1\ldots5} \overline{\eta}^{2}\right\},\\
\beta_5 &= b_3 = \left\{\psi^\mu, \chi^{56}, \omega^{1\ldots4}; \overline{\omega}^{1\ldots4}, \overline{\psi}^{1\ldots5}, \overline{\eta}^{3}\right\},\\
\beta_6 &= b_4 = \left\{\psi^\mu, \chi^{12}, y^{36}, \omega^{45}; \bar{y}^{36}, \overline{\omega}^{45}, \overline{\psi}^{1\ldots5}, \overline{\eta}^{1}\right\},\\
\beta_7 &= b_5 = \left\{\psi^\mu, \chi^{34}, y^{26}, \omega^{15}; \bar{y}^{26}, \overline{\omega}^{15}, \overline{\psi}^{1\ldots5}, \overline{\eta}^{2}\right\},
\end{aligned}
\tag{63}
$$

where included fermions are periodic, while all the rest are antiperiodic. One or two additional vectors and an appropriate set of GGSO phases are then used to break SO(10) and define each individual model. The first five vectors in Eq. (63) constitute the so-called NAHE set [90]. The basis vectors $\beta_1 = \zeta, \beta_2 = S$, and $\mathbb{1} = \zeta + b_1 + b_2 + b_3$ give rise to a $\mathcal{N} = 4$ supersymmetric model exhibiting $\mathrm{SO}(28) \times \mathrm{E}_8$ gauge symmetry with the vector ζ associated with the gauge group enhancement from $\mathrm{SO}(16)$ to E_8 . The vectors b_1 and b_2 correspond to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold twist that reduces space-time supersymmetry to $\mathcal{N} = 1$, the gauge symmetry group down to $\mathrm{SO}(10) \times \mathrm{SO}(6)^3 \times \mathrm{E}_8$, and gives rise to chiral fermions. Furthermore, the vectors b_4 and b_5 further break the $\mathrm{SO}(6)^3$ group factor to $\mathrm{SO}(4)^2 \times \mathrm{U}(1)$. In the notation used here, the "observable" $\mathrm{SO}(10)$ gauge symmetry is associated with $\overline{\psi}^{1\ldots5}$, while the "hidden" gauge group factor, E_8 , pertains to $\overline{\phi}^{1\ldots8}$.

其中包含的费米子是周期的，其余所有费米子都是反周期的。再利用 1 到 2 个额外矢量和一组合适的 GGSO 相位破缺 SO(10)，定义出每个独立模型。式 (63) 中的前五个矢量构成了所谓的 NAHE 集 [90]。基矢 $\beta_1=\zeta,\beta_2=S$ 和 $\mathbb{1}=\ \zeta+b_1+b_2+b_3$ 给出了一个 $\mathcal{N}=4$ 超对称模型，该模型具有 $\mathrm{SO}(28)\times\mathrm{E}_8$ 规范对称性，矢量 ζ 对应从 SO (16) 到 E_8 的规范群增强。矢量 b_1 和 b_2 对应 $\mathbb{Z}_2\times\mathbb{Z}_2$ 轨道面扭转，将时空超对称性约化为 $\mathcal{N}=1$，将规范群约化为 $\mathrm{SO}(10)\times\mathrm{SO}(6)^3\times\mathrm{E}_8$，并给出手征费米子。此外，矢量 b_4 和 b_5 进一步将 $\mathrm{SO}(6)^3$ 群因子破缺为 $\mathrm{SO}(4)^2\times\mathrm{U}(1)$。本文采用的标记中，“可观测”SO (10) 规范对称性与 $\overline{\psi}^{1\ldots5}$ 关联，而“隐藏”规范群因子 E_8 属于 $\overline{\phi}^{1\ldots8}$。

The flipped $\mathrm{SU}(5)\times\mathrm{U}(1)$ model is derived using the basis $\{\beta_1,\ldots,\beta_7,\beta_8\}$, where

翻转 $\mathrm{SU}(5)\times\mathrm{U}(1)$ 模型由基 $\{\beta_1,\ldots,\beta_7,\beta_8\}$ 构建得到，其中

$$\beta_8=\alpha=\left\{y^{46},\omega^{46};\bar{y}^{46},\overline{\omega}^{2346},\alpha\left(\overline{\psi}^{1\ldots5}\right)=\alpha\left(\overline{\eta}^{123}\right)=\alpha\left(\overline{\phi}^{1\ldots4}\right)=1/2,\overline{\phi}^{56}\right\}.$$

The additional vector α breaks the gauge symmetry down to $G=\mathrm{SU}(5)\times\mathrm{U}(1)'\times\ \mathrm{U}(1)^4\times\mathrm{SU}(4)\times\mathrm{SO}(10)$. Three fermion generations and a pair of $\mathrm{SU}(5)\times\mathrm{U}(1)'$ breaking Higgs fields in $\mathbf{10}_{1/2}+\overline{\mathbf{5}}_{-3/2}+\mathbf{1}_{5/2}$ and $\mathbf{10}_{1/2}+\overline{\mathbf{10}}_{-1/2}$ representations, respectively, arise from $b_1,\ldots,b_4$ and b_5 sectors. Four pairs of SU(5) vectors which accommodate the MSSM breaking Higgs fields come from the sectors 0 and $S+b_4+b_5$. The massless spectrum also comprises five hidden sector multiplets transforming as $(\mathbf{6},\mathbf{1})+(\mathbf{1},\mathbf{10})$ under $\mathrm{SU}(4)\times\mathrm{SO}(10)$ coming from the sectors $b_i+2\alpha\,(+\zeta)\,,i=1,\ldots,5$ and a number gauge singlets originating from 0 and $S+b_4+b_5$ sectors. In addition, six pairs of exotic fractional charge states in $\mathbf{4}_{\pm5/4}+\overline{\mathbf{4}}_{\mp5/4}$ representations arise from the sectors $b_1/b_4\pm\alpha\,(+\zeta)\,,b_1+b_3/b_4+\ b_5\pm\alpha\,(+\zeta)\,,S/b_1+b_2+b_4\pm\alpha\,(+\zeta)$.

额外的矢量 α 将规范对称性破缺为 $G=\mathrm{SU}(5)\times\mathrm{U}(1)'\times\ \mathrm{U}(1)^4\times\mathrm{SU}(4)\times\mathrm{SO}(10)$。三代费米子和一对分别在 $\mathbf{10}_{1/2}+\overline{\mathbf{5}}_{-3/2}+\mathbf{1}_{5/2}$ 和 $\mathbf{10}_{1/2}+\overline{\mathbf{10}}_{-1/2}$ 表示中实现破缺的 $\mathrm{SU}(5)\times\mathrm{U}(1)'$ 希格斯场，来源于 $b_1,\ldots,b_4$ 和 b_5 扇区。四对容纳最小超对称标准模型破缺希格斯场的 SU(5) 矢量来源于 0 扇区和 $S+b_4+b_5$ 扇区。无质量谱还包含五个隐藏区多重态，它们在 $\mathrm{SU}(4)\times\mathrm{SO}(10)$ 下按 $(\mathbf{6},\mathbf{1})+(\mathbf{1},\mathbf{10})$ 变换，来源于 $b_i+2\alpha\,(+\zeta)\,,i=1,\ldots,5$ 扇区，以及若干源自 0 扇区和 $S+b_4+b_5$ 扇区的规范单态。此外，六对处于 $\mathbf{4}_{\pm5/4}+\overline{\mathbf{4}}_{\mp5/4}$ 表示的奇异分数电荷态来源于 $b_1/b_4\pm\alpha\,(+\zeta)\,,b_1+b_3/b_4+\ b_5\pm\alpha\,(+\zeta)\,,S/b_1+b_2+b_4\pm\alpha\,(+\zeta)$. 扇区

The Pati-Salam [139] (PS) model is built upon the basis $\{\beta_1,\ldots,\beta_7,\beta_8,\beta_9\}$, where

帕蒂-萨拉姆 [139](PS) 模型基于基 $\{\beta_1,\ldots,\beta_7,\beta_8,\beta_9\}$ 构造，其中

$$\beta_8=b_6=\left\{y^6,\omega^6;\bar{y}^6,\overline{\omega}^6,\overline{\psi}^{1\ldots5},\overline{\eta}^{123},\overline{\phi}^{1\ldots4}\right\},$$

$$\beta_9=\alpha=\left\{y^{46},\omega^{46};\bar{y}^{46},\overline{\omega}^{2346},\overline{\psi}^{123},\overline{\eta}^{12},\overline{\phi}^{45}\right\},$$

break the gauge group to $G=\mathrm{SU}(4)\times\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}\times\mathrm{U}(1)^4\times\mathrm{U}(1)'\times\mathrm{SU}(8)$. The sectors $b_1,\ldots,b_5$ yield three chiral families and one pair of Pati-Salam [80] (PS) breaking Higgs multiplets transforming as $(\mathbf{4},\mathbf{2},\mathbf{1})+\left(\overline{\mathbf{4}},\mathbf{1},\mathbf{2}\right)$ and $(\mathbf{4},\mathbf{1},\mathbf{2})+\ \left(\overline{\mathbf{4}},\mathbf{1},\mathbf{2}\right)$, respectively, under $\mathrm{SU}(4)\times\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}$. Four pairs of MSSM Higgs doublets accommodated into bi-doublets $(\mathbf{1},\mathbf{2},\mathbf{2})$ accompanied with four pairs of triplets into the $(\mathbf{6},\mathbf{1},\mathbf{1})$ representation of the PS group come from the sectors $0,S+\ b_4+b_5$. The sectors $b_1\,(b_4)+\alpha,b_1+b_2\,(b_5)+b_4+$

$\alpha, b_2+b_3+b_5+\alpha$ give rise to ten pairs of exotic fractionally charged states transforming as $(\mathbf{1},\mathbf{2},\mathbf{1})+(\mathbf{1},\mathbf{1},\mathbf{2})$, while $S+b_2+b_4+\alpha$ provides a pair of exotic fourplets $(\mathbf{4},\mathbf{1},\mathbf{1})+\left(\overline{\mathbf{4}},\mathbf{1},\mathbf{1}\right)$. In addition, the massless spectrum includes five pairs of SU (8) vectors $\mathbf{8}+\overline{\mathbf{8}}$ from $b_i+b_6\,(+\zeta)$, $i=1,\ldots,4, b_2+b_3+b_5+b_6\,(+\zeta)$ and a number of non-abelian gauge group singlets from 0 and $S+b_4+b_5$ sectors.

将规范群破缺为 $G=\mathrm{SU}(4)\times\mathrm{SU}(2)_\mathrm{L}\times\mathrm{SU}(2)_\mathrm{R}\times\mathrm{U}(1)^4\times\mathrm{U}(1)'\times\mathrm{SU}(8)$。扇区 $b_1,\ldots,b_5$ 产生 3 个手征族和 1 对帕蒂-萨拉姆 [80](PS) 破缺希格斯多重态,在 $\mathrm{SU}(4)\times\mathrm{SU}(2)_\mathrm{L}\times\mathrm{SU}(2)_\mathrm{R}$ 下分别按 $(\mathbf{4},\mathbf{2},\mathbf{1})+\left(\overline{\mathbf{4}},\mathbf{1},\mathbf{2}\right)$ 和 $(\mathbf{4},\mathbf{1},\mathbf{2})+\left(\overline{\mathbf{4}},\mathbf{1},\mathbf{2}\right)$ 变换。4 对容纳在双二重态 $(\mathbf{1},\mathbf{2},\mathbf{2})$ 中的 MSSM 希格斯二重态,伴随 4 对位于 PS 群 $(\mathbf{6},\mathbf{1},\mathbf{1})$ 表示中的三重态,都来自扇区 $0, S+b_4+b_5$。扇区 $b_1\,(b_4)+\alpha, b_1+b_2\,(b_5)+b_4+\alpha, b_2+b_3+b_5+\alpha$ 产生 10 对按 $(\mathbf{1},\mathbf{2},\mathbf{1})+(\mathbf{1},\mathbf{1},\mathbf{2})$ 变换的奇特分数荷态对，而 $S+b_2+b_4+\alpha$ 提供 1 对奇特四元态对 $(\mathbf{4},\mathbf{1},\mathbf{1})+\left(\overline{\mathbf{4}},\mathbf{1},\mathbf{1}\right)$。此外，无质量谱包含来自 $b_i+b_6\,(+\zeta)$, $i=1,\ldots,4, b_2+b_3+b_5+b_6\,(+\zeta)$ 的 5 对 SU (8) 矢量 $\mathbf{8}+\overline{\mathbf{8}}$ ，以及来自 0 扇区和 $S+b_4+b_5$ 扇区的若干非阿贝尔规范群单态。

The Standard-like Model is described by the basis $\{\beta_1,\ldots,\beta_7,\beta_8,\beta_9\}$, with

类标准模型由基 $\{\beta_1,\ldots,\beta_7,\beta_8,\beta_9\}$ 描述，其中

$$\beta_8=\alpha=\left\{\psi^\mu,\chi^{12},y^{36}\omega^{45};\bar{y}^{36},\overline{\omega}^{45},\overline{\psi}^{1\ldots5},\overline{\eta}^{123},\overline{\phi}^{1\ldots3},\overline{\eta}^{12},\overline{\phi}^{1\ldots4}\right\}, \tag{64}$$

$$\beta_9=\beta=\left\{\psi^\mu,\chi^{34},y^{15},\omega^{26};\bar{y}^{1356},\overline{\omega}^{26},\alpha\left(\overline{\psi}^{1\ldots5}\right)=\alpha\left(\overline{\eta}^{123}\right)=\alpha\left(\overline{\phi}^{1567}\right)=1/2,\overline{\phi}^{34}\right\},$$

together with a specific set of GGSO phases [92]. The resulting gauge group is $G=\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)_\mathrm{C}\times\mathrm{U}(1)_\mathrm{L}\times\mathrm{SU}(2)^2\times\mathrm{SU}(3)\times\mathrm{U}(1)^4$. The sectors b_1, b_2 and b_3 produce three chiral generations of quarks and leptons, while three pairs of MSSM Higgs doublets arise from the sector 0. Moreover, the massless spectrum comprises a number of non-abelian gauge group singlets, some of which when acquiring vacuum expectation values (VEVs) could break $\mathrm{U}(1)_\mathrm{C}\times\mathrm{U}(1)_\mathrm{L}$ down to a linear combination that is identified with the weak hypercharge. Additional hidden sector $\mathrm{SU}(2)^2\times\mathrm{SU}(3)\times\mathrm{U}(1)^4$ states carrying only abelian MSSM gauge group charges arise from combinations of the vectors $\mathbb{1}, b_i, \alpha, \beta$. These include a number of fractional charge exotic multiplets. A variation of this model has been discussed in [62]. A different class of Standard-like Models not based on the NAHE basis set was discussed in [46].

以及一组特定的 GGSO 相 [92]。得到的规范群为 $G=\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)_\mathrm{C}\times\mathrm{U}(1)_\mathrm{L}\times\mathrm{SU}(2)^2\times\mathrm{SU}(3)\times\mathrm{U}(1)^4$ 。扇区 b_1, b_2 和 b_3 产生三夸克轻子手征代，而三对 MSSM 希格斯双态来自 0 扇区。此外，无质量谱包含多个非阿贝尔规范群单态，其中部分单态获得真空期望值 (VEV) 后可将 $\mathrm{U}(1)_\mathrm{C}\times\mathrm{U}(1)_\mathrm{L}$ 破缺为一个可被识别为弱超荷的线性组合。仅携带阿贝尔 MSSM 规范群荷的额外隐扇区 $\mathrm{SU}(2)^2\times\mathrm{SU}(3)\times\mathrm{U}(1)^4$ 态由向量 $\mathbb{1}, b_i, \alpha, \beta$ 的组合产生，其中包含若干分数电荷奇异多重态。该模型的一种变体已在文献 [62] 中讨论。文献 [46] 讨论了另一类不基于 NAHE 基组的类标准模型。

Among the main common characteristics of the aforementioned models is the presence of an anomalous abelian gauge symmetry. At first glance, several U(1)'s appear to be anomalous; however, after proper redefinitions, only a single linear combination, $\mathrm{U}(1)_A$, turns out to be anomalous, while all other orthogonal combinations are anomaly-free. The presence of an anomalous $\mathrm{U}(1)_A$ symmetry could destabilize the vacuum and lead to supersymmetry breaking unless it is cancelled via the Dine-Seiberg-Witten mechanism [51,54]. This involves VEVs for a set of charged fields φ_i that lead to $\mathrm{U}(1)_\mathrm{A}$ symmetry breaking and restabilize the vacuum at one loop. These VEVs should comply with the requirements of F - flatness and D -flatness conditions where the latter includes an anomalous abelian symmetry-related constraint of the form

上述模型的一个主要共同特征是存在反常阿贝尔规范对称性。乍看之下，多个 U(1) 似乎都是反常的；但经过恰当重新定义后，仅存在一个线性组合 $U(1)_A$ 是反常的，其余所有正交组合都无反常。反常 $U(1)_A$ 对称性的存在会破坏真空稳定性并引发超对称破缺，除非它通过 Dine-Seiberg-Witten 机制 [51,54] 被消除。该机制依赖一组带电场 φ_i 获得真空期望值，从而引发 $U(1)_A$ 对称性破缺，并在单圈水平重新稳定真空。这些真空期望值需要满足 F 平坦性和 D 平坦性条件，其中后者包含一个如下形式的、与反常阿贝尔对称性相关的约束:

$$D_{\rm A} = \sum_i q^i_{\rm A} |\varphi_i|^2 + \xi = 0, \;\; \text{with } \xi = \frac{1}{192\pi^2} \frac{2}{\alpha'} \operatorname{Tr} U(1)_{\rm A}. \tag{65}$$

This introduces an extra scale ξ which is typically one or two orders of magnitude below the string scale.

这引入了一个额外能标 ξ，其通常比弦标度低一到两个数量级。

The phenomenological analysis of the models described above entails finding a suitable solution of F - and D -flatness equations for the non-abelian singlet fields and, where applicable, the GUT symmetry breaking Higgs fields. Due to (65) flatness solutions lead to field VEVs of order ξ that usually break additional abelian gauge symmetries and provide masses for extra vector-like matter multiplets. The contribution of higher-order non-renormalizable superpotential terms should be also included in the flatness analysis when relevant. Particular attention must be paid on the requirement to keep a pair of MSSM Higgs multiplets light while providing sufficiently heavy masses to any additional color triplets that mediate nucleon decay. Another source of nucleon decay is baryon number violating operators coming from non-renormalizable superpotential interactions [59, 124, 135].

上述模型的唯象分析需要为非阿贝尔单态场以及 (适用情况下)GUT 对称性破缺希格斯场寻找 F 平坦性方程和 D 平坦性方程的合适解。根据 (65) 式，平坦性解给出数量级为 ξ 的场真空期望值，通常会破缺额外的阿贝尔规范对称性，并为额外类矢量物质多重态赋予质量。相关情况下，平坦性分析也应包含高阶非重整化超势项的贡献。必须特别注意满足以下要求: 保留一对轻 MSSM 希格斯多重态，同时给所有介导核子衰变的额外色三重态赋予足够大的质量。核子衰变的另一个来源是非重整化超势相互作用产生的重子数破坏算符 [59, 124, 135]。

Flipped SU(5) model phenomenology has been studied extensively in the literature [13, 59, 61, 116, 127, 128, 130, 133, 142] including the possibility of allowing hidden sector fields to develop VEVs [20,21,25,26]. The phenomenological aspects of the Pati-Salam model were discussed in [19, 132] and those of the Standard-like Models were analyzed in [39, 41 − 43, 62 − 68, 74, 89, 92] and also in [31-34]. In general, one finds flatness solutions that lead to hierarchical quark and lepton masses, with the third family being heavier than the other two, because third family masses come from trilinear superpotential terms, while the other two arise from (relatively suppressed) higher-order nonrenormalizable terms. The derivation of neutrino mass matrices is intricate, as neutrinos generally mix with all singlet fields in the model; however, in certain flatness solutions, neutrinos stay sufficiently light, thanks to a generalized seesaw mechanism [26,73].

翻转 SU(5) 模型的唯象学已在文献中得到广泛研究 [13, 59, 61, 116, 127, 128, 130, 133, 142]，其中包括允许隐层场获得真空期望值的研究 [20,21,25,26]。帕蒂-萨拉姆模型的唯象学方面已在 [19, 132] 中讨论，类标准模型的唯象学分析已见于 [39, 41 − 43, 62 − 68, 74, 89, 92] 以及 [31-34]。一般而言，我们可以得到会引发夸克与轻子质量分层的平坦解: 第三代夸克轻子重于另外两代，这是因为第三代质量来自三线性超势项，而另外两代的质量来自 (相对压低的) 高阶不可重整项。中微子质量矩阵的推导十分复杂，因为中微子通常会与模型中所有单态场混合；不过在特定平坦解中，得益于广义跷跷板机制 [26,73]，中微子可保持足够轻的质量。

A common feature of the models presented above is the presence of exotic fractionally charged particles in their massless spectra. Actually, the appearance of color-singlet fractionally charged states is a generic property of heterotic string compactifications based on level $k = 1$Kač-Moody algebra embeddings of the non-abelian group factors of the standard model (SM) [1, 144, 150]. Lacking any experimental confirmation and in view of strict cosmological constraints, one may wish to get rid of these states. This can be achieved if all fractionally charged states are confined. This scenario can be realized in the case of the flipped SU(5) model where all exotics transform nontrivially under an SU(4) hidden group factor [60,134]. Another interesting possibility is to project out all fractional charge exotics from the massless spectrum. As shown in [6, 30, 44] , this is possible in a class of Pati-Salam models (named "exophobic") built on the symmetric basis discussed in section "The Symmetric Basis and Model Scans".

上述模型的一个共同特征是，它们的无质量谱中存在奇异分数荷粒子。实际上，色单态分数荷态的出现，是基于标准模型 (SM) 非阿贝尔群因子的 $k = 1$Kač级卡茨-穆迪代数嵌入的杂化弦紧致化的一般性质 [1, 144, 150]。由于这类粒子尚未得到任何实验证实，且存在严格的宇宙学约束，人们可能希望消除这些态。如果所有分数荷态都被禁闭，就能实现这一点。这种情况可以在翻转 SU(5) 模型中实现: 该模型中所有奇异粒子在 SU(4) 隐群因子下按非平凡表示变换 [60,134]。另一种有趣的可能是将所有分数荷奇异粒子从无质量谱中投影出去。正如 [6, 30, 44] 中所示，这在一类帕蒂-萨拉姆模型 (称为“排奇模型”) 中是可以实现的，这类模型构建在“对称基与模型扫描”一节讨论的对称基之上。

Free fermionic models based on different gauge groups than those considered above have been discussed in [40,45,89,145]. Moreover, the possibility of building higher-level Kač-Moody algebra models using free fermions has been considered in [35, 52, 125] .

基于与上文不同规范群的自由费米子模型已在 [40,45,89,145] 中讨论。此外，利用自由费米子构建更高卡茨-穆迪代数层级模型的可能性已在 [35, 52, 125] 中得到研究。

The Symmetric Basis and Model Scans

对称基与模型扫描

An early attempt to classify heterotic string vacua in the FFF using computer-assisted search was reported in [76]. However, a systematic exploration and classification of FFF vacua became possible after adopting a new approach proposed in [78]. This is built on a 12-element basis

文献 [76] 报道了早期使用计算机辅助搜索对 FFF 中杂化弦真空进行分类的尝试。然而，在采用文献 [78] 提出的新方法后，才得以对 FFF 真空开展系统探索与分类。该方法建立在 12 元基的基础上

$$
\begin{aligned}
\beta_1 = \mathbb{1} &= \left\{\psi^\mu, \chi^{1\ldots6}, y^{1\ldots6}, \omega^{1\ldots6}; \bar{y}^{1\ldots6}, \overline{\omega}^{1\ldots6}, \overline{\eta}^{1,2,3}, \overline{\psi}^{1\ldots5}, \overline{\phi}^{1\ldots8}\right\},\\
\beta_2 = S &= \left\{\psi^\mu, \chi^{1\ldots6}\right\},\\
\beta_{2+i} = e_i &= \left\{y^i, \omega^i; \bar{y}^i, \overline{\omega}^i\right\}, i = 1\ldots6,\\
\beta_9 = b_1 &= \left\{x^{34}, \chi^{56}, y^{3456}; \bar{y}^{3456}, \overline{\psi}^{1\ldots5}, \overline{\eta}^1\right\},\\
\beta_{10} = b_2 &= \left\{\chi^{12}, x^{56}, y^{1256}; \bar{y}^{1256}, \overline{\psi}^{1\ldots5}, \overline{\eta}^2\right\},\\
\beta_{11} = z_1 &= \left\{\overline{\phi}^{1\ldots4}\right\},\\
\beta_{12} = z_2 &= \left\{\overline{\phi}^{5\ldots8}\right\},
\end{aligned}
\tag{66}
$$

which we refer to as symmetric basis in the sense that it treats the left/right internal fermionic coordinates, $y^{1\ldots6}/\bar{y}^{1\ldots6}$ and $\omega^{1\ldots6}/\overline{\omega}^{1\ldots6}$, symmetrically. The choice (66) also induces maximal rank reduction (by 6 units) and breaks gauge symmetry down to $G = \mathrm{SO}(10) \times \mathrm{U}(1)^3 \times \mathrm{SO}(8)^2$ for generic values of the GGSO. SO(10) spinors, 16(1 $\overline{\mathbf{6}}$), accommodating chiral matter, arise from the twisted sectors $\mathcal{S}^i_{pqrs} = S + b_i + pe_j + qe_k + re_\ell + se_m$ where $p, q, r, s = 0, 1$ and $(ijk\ell m)$ = (13456),(21256),(31234), with $b_3 = b_1 + b_2 + x, x = \mathbb{1} + S + e_1 + \ldots + e_6 + z_1 + z_2$. Similarly, SO(10) vectors, 10 carrying MSSM Higgs doublets come from the sectors $\mathcal{V}^i_{pqrs} = \mathcal{S}^i_{pqrs} + x$. Taking into account the fact that each of the aforementioned sectors $\mathcal{S}^i_{pqrs}, \mathcal{V}^i_{pqrs}$ can produce a single SO (10) spinor/vector multiplet, one can perform the GGSO projections explicitly and derive analytic formulae for the phenomenological characteristics of models, as the number of generations, the MSSM Higgs multiplets and the number of exotics, in terms of $c\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix}$. For example, the net chirality, n_g , that is, the number of chiral spinorial representations of SO (10) , n_{16} , minus the number of anti-spinorials, $n_{\overline{\mathbf{16}}}$, is given by

我们称之为对称基，因为它对左右内部费米坐标 $y^{1\ldots6}/\bar{y}^{1\ldots6}$ 和 $\omega^{1\ldots6}/\overline{\omega}^{1\ldots6}$ 进行对称处理。选择 (66) 还会实现最大秩约化 (降低 6 个单位)，并在 GGSO 的一般取值下将规范对称性破缺到 $G = \mathrm{SO}(10) \times \mathrm{U}(1)^3 \times \mathrm{SO}(8)^2$ 。容纳手征物质的 SO(10) 旋量 16(1 $\overline{\mathbf{6}}$) 来自扭曲 sector $\mathcal{S}^i_{pqrs} = S + b_i + pe_j + qe_k + re_\ell + se_m$ ，其中 $p, q, r, s = 0, 1$ 和 $(ijk\ell m)$ = 为 (13456)、(21256)、(31234)，满足 $b_3 = b_1 + b_2 + x, x = \mathbb{1} + S + e_1 + \ldots + e_6 + z_1 + z_2$ 。类似地，携带 MSSM 希格斯二重态的 SO(10) 向量 10 来自 sector $\mathcal{V}^i_{pqrs} = \mathcal{S}^i_{pqrs} + x$ 。考虑到上述每个 sector $\mathcal{S}^i_{pqrs}, \mathcal{V}^i_{pqrs}$ 只能产生一个 SO (10) 旋量/向量多重态，我们可以显式执行 GGSO 投影，并以 $c\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix}$ 为参数推导出模型唯象特征 (如代数量、MSSM 希格斯多重态数量和外态数量) 的解析公式。例如，净手征性 n_g 即 SO (10) 的手征旋量表示 n_{16} 的数量减去反手征旋量表示 $n_{\overline{\mathbf{16}}}$ 的数量，由下式给出

$$n_g = n_{16} - n_{\overline{16}} = \frac{1}{2^4} \sum_{p,q,r,s\in\{0,1\}} \sum_{i=1}^{3} c\begin{bmatrix} \mathcal{S}^i{}_{\text{prrs}} \\ (irs) \end{bmatrix}^* \times$$

(67)

$$\prod_{i'=2i-1,2i} \left(1 - c\begin{bmatrix} \mathcal{S}^i_{pqrs} \\ e_{i'} \end{bmatrix}^*\right) \prod_{k'=1,2} \left(1 - c\begin{bmatrix} \mathcal{S}^i_{pqrs} \\ z_{k'} \end{bmatrix}^*\right),$$

where we use the following notation for the chirality factor $c\begin{bmatrix} \mathcal{S}^i{}_{\text{pqrs}} \\ (irs) \end{bmatrix}$: $(1rs) = S + \ b_2 + (1-r)\,e_5 + (1-s)\,e_6, (2rs) = (S + b_1 + (1-r)\,e_5 + (1-s)\,e_6$, and $(3rs) = \ (S + b_1 + (1-r)\,e_3 + (1-s)\,e_4$. Similarly, for the vectorial representations of $SO\,(10)$, we have

其中我们对手征因子 $c\begin{bmatrix} \mathcal{S}^i{}_{\text{pqrs}} \\ (irs) \end{bmatrix}$: $(1rs) = S + \ b_2 + (1-r)\,e_5 + (1-s)\,e_6, (2rs) = (S + b_1 + (1-r)\,e_5 + (1-s)\,e_6$ 和 $(3rs) = \ (S + b_1 + (1-r)\,e_3 + (1-s)\,e_4$ 使用如下记号。类似地，对于 $SO\,(10)$ 的向量表示，我们有

$$n_{10} = \frac{1}{2^4} \sum_{i=1}^{3} \sum_{p,q,r,s\in\{0,1\}} \prod_{i'=2i-1,2i} \left(1 - c\begin{bmatrix} \mathcal{V}^i_{pqrs} \\ e_{i'} \end{bmatrix}^*\right) \prod_{k'=1,2} \left(1 - c\begin{bmatrix} \mathcal{V}^i_{pqrs} \\ z_{k'} \end{bmatrix}^*\right).$$

(68)

The introduction of one or two additional basis vectors, involving the 16 right complex fermions solely, further breaks gauge symmetry and truncates SO(10) spinors/vectors appropriately. In this case, the above equations are modified accordingly and extended . Although intricate, and not invertible in the sense that one cannot solve analytically for the projection coefficients in terms of the model data, e.g., n_g, n_{10} , these formulae can be readily evaluated using a computer code and utilized to efficiently scan big classes of FFF vacua.

引入仅涉及 16 个右复费米子的一到两个额外基矢量，会进一步破缺规范对称性，并对 SO(10) 旋量/矢量进行恰当截断。这种情况下，上述方程会得到相应修改与延拓。尽管这些公式较为复杂，且无法做到可逆——即不能以模型数据例如 n_g, n_{10} 解析求解投影系数，但可通过计算机代码轻松计算，并用于高效扫描大类 FFF 真空。

The method outlined above was introduced in [107] in the context of classification of type IIA/B string models and has been further developed and employed in [78] to classify heterotic SO(10) vacua. It was also used in [6,7] to classify a huge collection of 10^{11} Pati-Salam string models, generated by the introduction of a single additional basis vector $\beta_{13} = \alpha = \left\{\bar{\phi}^{45}, \bar{\phi}^{12}\right\}$. It was shown that one in a billion Pati-Salam vacua possesses three generations together with the required gauge symmetry breaking Higgs fields and is free of massless fractional charged exotics (exophobic models). Big collections of flipped SU(5), Standard-like Models, and left-right symmetric models have been also studied and categorized in [99,100], and [75], respectively. This framework can be employed to incorporate additional constraints, related to superpotential couplings, in model scans, as the existence of tree-level top quark mass related coupling in the effective superpotential that naturally leads to a large mass for the top quark [44, 140].

上述方法最早在文献 [107] 中针对 IIA/B 型弦模型分类提出，后在文献 [78] 中得到进一步发展，被用于杂化 SO(10) 真空分类。文献 [6,7] 也用该方法对引入单个额外基矢量 $\beta_{13} = \alpha = \left\{\bar{\phi}^{45}, \bar{\phi}^{12}\right\}$ 生成的大量 10^{11} 帕蒂-萨拉姆弦模型进行了分类。研究发现，十亿分之一的帕蒂-萨拉姆真空同时满足三个代的要求、包含所需规范对称性破缺希格斯场，且不存在无质量分数荷外粒子 (厌外模型)。文献 [99,100] 和 [75] 还分别对大集合的翻转 SU(5)、类标准模型和左右对称模型做了研究与分类。该框架可用于在模型扫描中纳入超势耦合相关的额外约束，例如有效超势中树级顶夸克质量相关耦合，该耦合自然能给顶夸克带来大质量 [44, 140]。

The class of models generated by the symmetric basis (66) has been shown to exhibit an interesting symmetry called spinor-vector duality. This refers to the interchange of matter representations of the SO(10) group factor16(spinorial) and10(vectorial) accompanied by the1(singlet) that appear in the decomposition $\mathbf{27} = (\mathbf{16}, 1/2) + (\mathbf{10}, -1) + (\mathbf{1}, +2)$ of $\mathrm{E}_6 \supset \mathrm{SO}(10)$ [16, 47, 69, 72, 81, 82] . This symmetry has been employed in [95] to construct Pati-Salam models with an extra family universal Z' symmetry that could survive down to low energies.

对称基 (66) 生成的模型类被证明具有一种有趣的对称性, 称为旋量-矢量对偶性。该对称性指 SO(10) 群因子的物质表示 16(旋量表示) 与 10(矢量表示) 发生互换，同时伴随出现在 $\mathbf{27} = (\mathbf{16}, 1/2) + (\mathbf{10}, -1) + (\mathbf{1}, +2)$ 对 $\mathrm{E}_6 \supset \mathrm{SO}(10)$ [16, 47, 69, 72, 81, 82] 分解中的 1(单态) 也发生互换。文献 [95] 利用该对称性构造了带有额外族普适 Z' 对称性的帕蒂-萨拉姆模型，该对称性可以一直保持到低能标。

The symmetric basis can also be utilized for the efficient implementation of the FFF in the exploration of the string landscape using advanced computational methods, such as genetic algorithms [24], satisfiability modulo theory [93] and, recently, on quantum annealers [22].

对称基还可配合先进计算方法，用于高效实现 FFF，以探索弦景观，例如遗传算法 [24]、可满足性模理论 [93]，近年还应用于量子退火器 [22]。

Non-supersymmetric Models

非超对称模型

As has been known since the early days of the first superstring revolution, besides the space-time supersymmetric $\mathrm{E}_8 \times \mathrm{E}_8/\mathrm{SO}(32)$ heterotic string theory [106], one can construct consistent non-supersymmetric theories as the $\mathrm{SO}(16) \times \mathrm{SO}(16)$ heterotic string model [17,50]. However, string phenomenology has mainly focused on $\mathcal{N} = 1$ supersymmetric models for two reasons. The first is the presence of tachyons in the physical spectrum of a generic non-supersymmetric theory signaling vacuum instability and the second is the appearance of large one-loop dilaton tadpoles and their back-reaction on the classical vacuum. However, the lack of evidence in favor of supersymmetry in recent experiments motivates the exploration of non-supersymmetric string vacua.

自第一次超弦革命早期人们就已知，除了时空超对称 $E_8 \times E_8/SO(32)$ 杂化弦理论 [106] 之外，还可以构造出自洽的非超对称理论，比如 $SO(16) \times SO(16)$ 杂化弦模型 [17,50]。但弦唯象学主要聚焦于 $\mathcal{N} = 1$ 超对称模型，原因有二: 其一，一般非超对称理论的物理谱中存在快子，预示真空不稳定；其二，会出现大的一圈 dilaton 蝌蚪图及其对经典真空的反作用。不过，近年实验中没有发现支持超对称存在的证据，推动了人们对非超对称弦真空的探索。

It is not difficult to construct four-dimensional non-supersymmetric string models in the FFF framework. It amounts to choosing the spin-structure coefficient $c\begin{bmatrix} S \\ \beta \end{bmatrix} = +1$ for some basis vector $\beta \cap S = \varnothing$ so as to project out the gravitino state arising in the sector $S = \{\psi^\mu, \chi^1, \ldots, \chi^6\}$. One may then eliminate tachyons from the physical spectrum by an appropriate choice of GGSO coefficients $c\begin{bmatrix} \beta_i \\ \beta_j \end{bmatrix}$. Several models have been constructed along these lines [36,84-87,103,104]. Unfortunately, this direct method generally results in explicit supersymmetry breaking at the string scale. Another interesting possibility is the spontaneous breaking of supersymmetry via coordinate-dependent compactifications [80, 120, 123, 141] , corresponding to a stringy realization of the Scherk-Schwarz mechanism [146, 147]. In its simplest form, this can be achieved by picking an extra dimension X^5 compactified on a circle of radius R and imposing nontrivial monodromies $\Phi\left(X^5 + 2\pi R\right) = e^{iQ}\Phi\left(X^5\right)$ around the circle, so that the states Φ of the theory are periodic only up to the action of a symmetry generator Q . This results in a shift of the tower of Kaluza-Klein (KK) masses of the charged states. Identifying the symmetry operator e^{iQ} with the fermion number parity $(-1)^F$ leads to spontaneous supersymmetry breaking at a scale $m_{3/2} \sim 1/R$. In general, the scalar potential of the resulting theories is no longer super-protected, and quantum corrections will typically generate a non-trivial cosmological constant. In the large radius limit, the one-loop effective potential takes the form [23,112]

在 FFF 框架中构造四维非超对称弦模型并不困难，只需对某些基矢 $\beta \cap S = \varnothing$ 选择自旋结构系数 $c\begin{bmatrix} S \\ \beta \end{bmatrix} = +1$，从而投影掉 sector $S = \{\psi^\mu, \chi^1, \ldots, \chi^6\}$ 中产生的引力微子态。之后可以通过恰当选择 GGSO 系数 $c\begin{bmatrix} \beta_i \\ \beta_j \end{bmatrix}$ 从物理谱中消除快子，沿着这一思路已经构造出了多个模型 [36,84-87,103,104]。遗憾的是，这种直接方法一般会导致超对称在弦尺度处显式破缺。另一种有趣的可能是通过依赖坐标的紧化 [80, 120, 123, 141] 实现超对称自发破缺，这对应 Scherk-Schwarz 机制的弦论实现 [146,147]。在最简单的形式中，这可以通过额外维 X^5 紧化在半径为 R 的圆上，并对圆施加非平庸单值群 $\Phi\left(X^5 + 2\pi R\right) = e^{iQ}\Phi\left(X^5\right)$ 实现，此时理论的态 Φ 仅在对称生成元 Q 的作用下才具有周期性。这会导致带电站的卡鲁扎-克莱因 (KK) 质量谱发生平移。将对称算符 e^{iQ} 与费米子数宇称 $(-1)^F$ 对应，就会在能标 $m_{3/2} \sim 1/R$ 处产生超对称自发破缺。一般而言，所得理论的标量势不再具有超保护，量子修正通常会产生非平庸宇宙学常数。在大半径极限下，一圈有效势的形式为 [23,112]

$$V_{\text{eff}} = -\frac{\zeta}{R^4} + O\left(e^{-\lambda R}\right), \tag{69}$$

where $\zeta \sim n_B - n_F$ is proportional to the massless spectrum degeneracies and λ is a positive constant of order one. Unfortunately, this value for the cosmological constant deviates from its observed one by several orders of magnitude, even if we were to lower the compactification scale down to the TeV range. However, the leading contribution in (69) may vanish for models where the number n_F of the massless fermionic degrees of freedom equals the number n_B of bosonic massless states, namely, $n_B = n_F$. In this special class of models, termed super-no-scale models in [121], the cosmological constant is exponentially suppressed for sufficiently large values of the compactification radius. Such non-supersymmetric models have recently attracted attention [8, 27, 110, 111, 122] in the context of string phenomenology. Concrete super-no-scale models with an even number of generations were constructed and discussed in [2, 8, 9, 96, 97, 101] , while three generation models with exponentially suppressed cosmological constant were also shown to be possible [102], provided additional constraints are satisfied, at least as far as $\mathbb{Z}_2 \times \mathbb{Z}_2$ compactifications are concerned.

其中 $\zeta \sim n_{\mathrm{B}} - n_{\mathrm{F}}$ 正比于无质量谱的简并度，λ 是一个量级为 1 的正常数。遗憾的是，该宇宙学常数的取值与观测值相差数个数量级，即便我们将紧致化尺度降低到 TeV 范围也是如此。但对于无质量费米自由度数量 n_{F} 等于无质量玻色态数量 n_{B} (即 $n_{\mathrm{B}} = n_{\mathrm{F}}$) 的模型，式 (69) 中的领头阶贡献可以为零。在文献 [121] 中这类特殊模型被称为超无标度模型，对于足够大的紧致化半径，这类模型的宇宙学常数会被指数压低。这类非超对称模型近来在弦唯象学领域引发了关注 [8, 27, 110, 111, 122]。[2, 8, 9, 96, 97, 101] 中已经构造并讨论了带有偶数代的具体超无标度模型，同时文献 [102] 也表明，只要满足额外约束，存在带三代且宇宙学常数被指数压低的模型也是可能的，至少就 $\mathbb{Z}_2 \times \mathbb{Z}_2$ 紧致化而言确实如此。

The stringy version of the Scherk-Schwarz mechanism can be conveniently realized in terms of freely acting $\mathbb{Z}_2$ orbifolds with action $g = (-1)^F \delta$, where δ is an order-2 translation (shift) along a nontrivial cycle of the compactification space [79]. A large class of semi-realistic FFF models may be recast into an orbifold representation and deformed away from the fermionic point by marginal operators [83, 96, 97, 101] . This effectively reinstates the dependence on the compactification moduli and allows one to study the conditions under which the supersymmetry breaking is spontaneous (Scherk-Schwarz type) or explicit. The procedure for this map is outlined in the following section and illustrated with an explicit example.

谢尔克-施瓦茨机制的弦论版本可以方便地通过自由作用的 $\mathbb{Z}_2$ 轨形实现，其作用为 $g = (-1)^F \delta$ ，其中 δ 是沿紧致化空间一个非平凡闭链的二阶平移 (平移移动)[79]。一大类半现实的 FFF 模型可以改写为轨形表示，并通过边缘算符 [83, 96, 97, 101] 偏离费米点形变。这有效恢复了模型对紧致化模空间的依赖，使我们可以研究超对称破缺为自发 (谢尔克-施瓦茨型) 或显式破缺的条件。该映射的步骤将在下一节概述，并通过一个具体示例说明。

Map to Orbifolds

映射到轨形

The fermionic models we have discussed live in special points of moduli space, where the compactification moduli take fixed values compatible with bosonization. However, in many applications, it is necessary to reinstate the moduli dependence of the theory, for instance, in order to study the dependence of string threshold corrections to the compactification moduli [97]. Provided the corresponding moduli scalars are not projected out of the string spectrum, and provided their vertex operators are exactly marginal, they can be used to marginally deform the theory away from the fermionic point. This can be most easily accomplished if the free fermionic models are mapped to toroidal orbifolds with appropriate $(\mathbb{Z}_2)^M$ rotations (or even translations) on the internal space (super)coordinates. It should be mentioned that considerable efforts have been made in the literature to bridge the gap between the FFF, orbifolds, and geometric formulations and obtain a unified treatment, cf. [14, 37, 56, 83, 91, 96, 97, 101]. The method for the map we present here appeared in its early form in [83] and further developed in [98].

我们讨论过的费米子模型位于模空间的特殊点上，在这些点上，紧致化模取与玻色化相容的固定值。但在许多应用中，必须恢复理论对模的依赖关系，例如，用来研究弦阈修正对紧致化模的依赖关系 [97]。只要对应的模标量没有从弦谱中被投影出去，且它们的顶点算子是恰好边缘的，就可以用它们对理论做边缘形变，使其偏离费米点。如果将自由费米子模型映射到对内部空间 (超) 坐标做适当 $(\mathbb{Z}_2)^M$ 旋转 (甚至平移) 的环面轨形，这个过程会最容易实现。需要说明的是，文献中已经付出了大量努力来弥合 FFF、轨形和几何表述之间的鸿沟，并得到统一的处理，参见 [14, 37, 56, 83, 91, 96, 97, 101]。我们在此介绍的映射方法，其早期形式出现于文献 [83]，并在文献 [98] 中得到进一步发展。

For simplicity, consider two bosonic coordinates X^1, X^2 compactified on a T^2 , without any orbifold rotation. They can be fermionized in terms of four (auxiliary) real left-moving fermions $y^1, \omega^1, y^2, \omega^2$, as $\sqrt{2/\alpha'}\partial X^i = y^i\omega^i$ and similarly for the right-movers. Generalizing Eq. (17), we pick identical boundary conditions for the four fermions and sum the contribution to the partition function over all spin structures. After appropriate shifts of the summation variables, one obtains

为简单起见，考虑两个紧致在 T^2 上的玻色坐标 X^1, X^2 ，没有任何轨形旋转。它们可以用四个 (辅助) 实左行费米子 $y^1, \omega^1, y^2, \omega^2$ 费米化，形式为 $\sqrt{2/\alpha'}\partial X^i = y^i\omega^i$ ，右行费米子同理。推广方程 (17)，我们为四个费米子选取相同的边界条件，并对所有自旋结构求和配分函数的贡献。对求和变量做适当移位后，可以得到

$$\frac{1}{2}\sum_{\gamma,\delta=0,1}\frac{\vartheta^2\begin{bmatrix}\gamma\\\delta\end{bmatrix}\bar{\vartheta}^2\begin{bmatrix}\gamma\\\delta\end{bmatrix}}{\eta^2\bar{\eta}^2} = \frac{1}{\eta^2\bar{\eta}^2}\Gamma_{2,2}\left(\mathrm{i}\alpha', \mathrm{i}\right), \tag{70}$$

where $\Gamma_{2,2}(T, U)$ is the partition function of the (2,2) Narain lattice

其中 $\Gamma_{2,2}(T, U)$ 是 (2,2)Narain 格的配分函数

$$\Gamma_{2,2}(T, U) = \sum_{m_i, n_i \in \mathbb{Z}} q^{\frac{\alpha'}{4}|P_L|^2}\bar{q}^{\frac{\alpha'}{4}|P_R|^2}, \tag{71}$$

with the complexified lattice momenta defined as

复化格动量定义为

$$P_L = \frac{m_2 - Um_1 + T(n_1 + Un_2)}{\sqrt{T_2U_2}}, P_R = \frac{m_2 - Um_1 + \bar{T}(n_1 + Un_2)}{\sqrt{T_2U_2}}. \tag{72}$$

Here, T and U are the Kahler and complex structure moduli of the T^2 , respectively. Notice that the fermionization of the bosonic T^2 coordinates occurs only for a square lattice, as indicated by the purely imaginary values $T = \mathrm{i}\alpha'$ and $U = \mathrm{i}$ in (70). Permissible orbifold actions compatible with the fermionization must act crystallographically on this square lattice, so that $\mathbb{Z}_2$ and $\mathbb{Z}_4$ actions can be realized. This naturally generalizes to orbifolds on higher-dimensional tori, as well as to orbifolds involving both twists (rotations) and shifts (translations).

此处，T 和 U 分别是 T^2 的 Kahler 模和复结构模。注意，只有正方格才能对玻色 T^2 坐标做费米化，(70) 式中 $T = \mathrm{i}\alpha'$ 和 $U = \mathrm{i}$ 的纯虚值就体现了这一点。与费米化相容的容许轨形作用必须在这个正方格上结晶作用，这样才能实现 $\mathbb{Z}_2$ 和 $\mathbb{Z}_4$ 作用。这自然可以推广到高维环面的轨形，也可以推广到同时包含扭转 (旋转) 和平移 (平移) 的轨形。

Let us assume a simple $\mathbb{Z}_2$ twist under which the bosonic coordinates X^1, X^2 of T^2 are rotated by an angle π , such that $X^1 \to -X^1$ and $X^2 \to -X^2$. In terms of the free fermions, this action is reproduced by the twist

我们假设一个简单的 $\mathbb{Z}_2$ 扭转，在该扭转下，T^2 的玻色坐标 X^1, X^2 被旋转角度 π ，满足 $X^1 \to -X^1$ 和 $X^2 \to -X^2$ 。用自由费米子表述的话，这个作用可以通过如下扭转重现

$$y^1 \to -y^1 \, , y^2 \to -y^2 \, , \omega^1 \to \omega^1 \, , \omega^2 \to \omega^2, \tag{73}$$

with an identical action on the right-movers. Furthermore, in order to preserve $\mathcal{N} = 1$ worldsheet supersymmetry, the orbifold action on the bosonic coordinates X^1, X^2 must be accompanied by a simultaneous rotation of their real (left-moving) fermion superpartners, i.e., $\psi^1 \to -\psi^1$ and $\psi^2 \to -\psi^2$. Of course, this $\mathbb{Z}_2$ action on a single T^2 does not preserve any space-time supercharges. The situation can be remedied by extending the $\mathbb{Z}_2$ action to an additional 2-torus, so that one considers $T^4/\mathbb{Z}_2$.

对右行费米子有相同的作用。此外，为了保持世界面上的 $\mathcal{N} = 1$ 超对称，对玻色坐标 X^1, X^2 做的轨形作用必须同时伴随对它们的实 (左行) 费米超伙伴的旋转，即 $\psi^1 \to -\psi^1$ 和 $\psi^2 \to -\psi^2$ 。当然，这个作用在单个 T^2 上的 $\mathbb{Z}_2$ 作用不保持任何时空超荷。我们可以通过将 $\mathbb{Z}_2$ 作用拓展到额外的二维环面来解决这个问题，即我们考虑 $T^4/\mathbb{Z}_2$ 。

We postpone the discussion of the full supersymmetry-preserving orbifold action for later and instead focus on the contribution of a single T^2 factor to the partition function in the case $T/\alpha' = U = \mathrm{i}$. This amounts to computing the genus 1 path integral of the compact bosonic coordinates spanning $T^2/\mathbb{Z}_2$. Upon conveniently complexifying $Z = X^1 + \mathrm{i}X^2$, and introducing the $\mathbb{Z}_2$ orbifold parameters $h, g \in \{0, 1\}$ encoding the twists of the (initially periodic) boundary conditions of Z along the two cycles of the worldsheet torus, the boundary conditions read

我们将保留全部超对称的完整轨形作用的讨论推迟到后面，转而聚焦于 T^2 单个因子对情况 $T/\alpha' = U = \mathrm{i}$ 下配分函数的贡献。这相当于计算张成 $T^2/\mathbb{Z}_2$ 的紧致玻色坐标的亏格 1 路径积分。在对 $Z = X^1 + \mathrm{i}X^2$ 做适当复化，并引入编码了 Z 在世界面 torus 两个闭链上 (最初周期) 边界条件扭转的 $\mathbb{Z}_2$ 轨形参数 $h, g \in \{0, 1\}$ 后，边界条件可写为

$$Z\left(\sigma^1 + 1, \sigma^2\right) = \mathrm{e}^{\mathrm{i}\pi h} Z\left(\sigma^1, \sigma^2\right),$$

$$Z\left(\sigma^1, \sigma^2 + 1\right) = \mathrm{e}^{\mathrm{i}\pi g} Z\left(\sigma^1, \sigma^2\right). \tag{74}$$

The general mode expansion compatible with these boundary conditions reads

与这些边界条件相容的一般模展开为

$$Z\left(\sigma^1,\sigma^2\right) = z_0 + Q\sigma^1 + \widetilde{Q}\sigma^2 + \sum_{m,n\in\mathbb{Z}} Z_{n,m}\mathrm{e}^{2\pi\mathrm{i}\left[(n+h/2)\sigma^1+(m+g/2)\sigma^2\right]}, \tag{75}$$

where σ^1, σ^2 are the 2D coordinates parametrizing the worldsheet torus, z_0 is the (center of mass) zero mode, and Q and $\widetilde{Q}$ are (complexified) windings corresponding to the classical solution $\square Z = 0$, while $Z_{n,m}$ correspond to the (quantum) oscillator modes.

其中 σ^1, σ^2 是参数化世界面环面的二维坐标，z_0 是 (质心) 零模，Q 和 $\widetilde{Q}$ 是对应经典解 $\square Z = 0$ 的 (复化) 绕数，而 $Z_{n,m}$ 对应 (量子) 振荡模。

Clearly, in the sector $h = g = 0$ where Z is untwisted under both cycles and the boundary conditions trivialize, one recovers the r.h.s. of (70). Indeed, in this case, the path integral over the oscillator modes simply produces Dedekind functions $\left(\eta^2\bar{\eta}^2\right)^{-1}$, the classical BPS solution produces the Narain lattice sum in the Lagrangian representation, and the integral over the zero mode z_0 produces the appropriate T^2 volume factor. After Poisson resumming over the $\widetilde{Q}$ windings, the volume factor cancels and one recovers the familiar Hamiltonian form of the Narain lattice (71), in terms of the complexified lattice momenta P_L, P_R of Eq. (72).

显然，在两个闭链下 Z 都未扭转、边界条件平庸的扇区 $h = g = 0$ 中，我们可以回收到 (70) 的右侧。实际上在该情况下，振荡模的路径积分恰好给出戴德金函数 $\left(\eta^2\bar{\eta}^2\right)^{-1}$ ，经典 BPS 解给出拉格朗日表示下的 Narain 晶格和，零模 z_0 的积分给出合适的 T^2 体积因子。对 $\widetilde{Q}$ 绕数做泊松重求和后，体积因子抵消，我们回收到式 (72) 中复化晶格动量 P_L, P_R \$ 下，大家熟知的 Narain 晶格哈密顿形式 (71)。

The contribution of the twisted sectors $(h, g) \neq (0, 0)$ is more involved. First, substituting the mode expansion (75) into the boundary conditions (74) requires $\left(1 - \mathrm{e}^{\mathrm{i}\pi h}\right) Q = \left(1 - \mathrm{e}^{\mathrm{i}\pi g}\right) Q = 0$ and similarly for $\widetilde{Q}$, implying the absence of windings in the twisted sectors, $Q = \widetilde{Q} = 0$. One is left to impose the center of mass conditions

扭转扇区 $(h, g) \neq (0, 0)$ 的贡献更复杂。首先，将模展开 (75) 代入边界条件 (74) 要求 $\left(1 - \mathrm{e}^{\mathrm{i}\pi h}\right) Q = \left(1 - \mathrm{e}^{\mathrm{i}\pi g}\right) Q = 0$ ，对 $\widetilde{Q}$ 也同理，这意味着扭转扇区中不存在绕数，即 $Q = \widetilde{Q} = 0$ 。接下来需要施加质心条件

$$\left(1 - \mathrm{e}^{\mathrm{i}\pi h}\right) z_0 = 0 \left(\bmod \Lambda\right),$$

(76)

$$\left(1 - \mathrm{e}^{\mathrm{i}\pi g}\right) z_0 = 0 \left(\bmod \Lambda\right),$$

where $\bmod \Lambda$ simply denotes the fact that the above conditions should be satisfied modulo T^2 lattice vectors. In other words, the center of mass mode z_0 is restricted to lie on the simultaneous fixed points under both h and g twists. Let us now return to the path integral. Integrating out the oscillator modes is straightforward and amounts to computing the zeta-regularized determinant $\mathrm{Det}'_{h,g}\square$ of the worldsheet torus Laplacian under the twisted boundary conditions (h, g) , yielding $\left|\eta/\vartheta\begin{bmatrix}1-h\\1-g\end{bmatrix}\right|^2$. Since $Q = \widetilde{Q} = 0$, there is no BPS lattice sum present. Moreover, there is no volume factor, but we instead have to perform a discrete sum over the zero mode z_0 , which spans the simultaneous orbifold fixed points in the (h, g) sector. For $T^2/\mathbb{Z}_2$ with

$z_0 \sim z_0+1 \sim z_0+\mathrm{i}$, the four fixed points are given by $z_0 \in \{0, 1/2, \mathrm{i}/2, (1+\mathrm{i})/2\}$. Putting everything together, the contribution to the partition function of the compact scalars X^1, X^2 parametrizing $T^2/\mathbb{Z}_2$ in the (h,g) orbifold sector reads

其中 mod Λ 仅表示上述条件需对模 T^2 格矢成立。换句话说，质心模 z_0 被限制在同时处于 h 扭转和 g 扭转下的不动点上。现在回到路径积分: 对振荡模的积分十分直接，相当于计算世界面环面拉普拉斯算子在扭曲边界条件 (h,g) 下的 zeta 正则化行列式 $\mathrm{Det}'_{h,g}\Box$ ，结果为 $\left|\eta/\vartheta\begin{bmatrix}1-h\\1-g\end{bmatrix}\right|^2$ 。由于 $Q=\widetilde{Q}=0$ ，不存在 BPS 格和。此外，这里没有体积因子，但我们需要对零模 z_0 进行离散求和，该零模张成 (h,g) 扇区中同时存在的轨形不动点。对于满足 $z_0 \sim z_0+1 \sim z_0+\mathrm{i}$ 的 $T^2/\mathbb{Z}_2$ ，四个不动点由 $z_0 \in \{0, 1/2, \mathrm{i}/2, (1+\mathrm{i})/2\}$ 给出。综上，参数化 $T^2/\mathbb{Z}_2$ 的紧致标量 X^1, X^2 对 (h,g) 轨形扇区配分函数的贡献为

$$\frac{1}{\eta^2\bar{\eta}^2}\Gamma_{2,2}\begin{bmatrix}h\\g\end{bmatrix}(\mathrm{i}\alpha',\mathrm{i}) = \frac{1}{\eta^2\bar{\eta}^2}\begin{cases}\Gamma_{2,2}(\mathrm{i}\alpha',\mathrm{i}), (h,g)=(0,0)\\ \left|\frac{2\eta^3}{\vartheta\begin{bmatrix}\frac{1-h}{1-g}\end{bmatrix}}\right|^2, (h,g)\neq(0,0)\end{cases}. \tag{77}$$

From the orbifold perspective, the parameter $h=0,1$ labels the orbifold twisted sectors, while summation over $g=0,1$ imposes the $\mathbb{Z}_2$ projection.

从轨形的角度看，参数 $h=0,1$ 标记轨形扭曲扇区，对 $g=0,1$ 求和则实现了 $\mathbb{Z}_2$ 投影。

It is now straightforward to see that the same result (Eq. (77)) can be equivalently obtained from the free fermion system $y^i, \omega^i, \bar{y}^i, \bar{\omega}^i$ with the twisted boundary conditions (73). Indeed, assume that before the twists, all auxiliary fermions carried the same boundary conditions (γ,δ) as in (70). Introducing the twist implies the replacement

现在可以轻易看出，相同结果 (式 (77)) 也可以等价地从带扭曲边界条件 (73) 的自由费米子系统 $y^i, \omega^i, \bar{y}^i, \bar{\omega}^i$ 得到。事实上，假设扭转前所有辅助费米子都满足和 (70) 中相同的边界条件 (γ,δ) ，引入扭转就意味着替换

$$\frac{1}{2}\sum_{\gamma,\delta=0,1}\frac{\vartheta^2\begin{bmatrix}\gamma\\\delta\end{bmatrix}\bar{\vartheta}^2\begin{bmatrix}\gamma\\\delta\end{bmatrix}}{\eta^2\bar{\eta}^2} \to \frac{1}{2}\sum_{\gamma,\delta=0,1}\frac{\vartheta\begin{bmatrix}\gamma\\\delta\end{bmatrix}\vartheta\begin{bmatrix}\gamma+h\\\delta+g\end{bmatrix}\bar{\vartheta}\begin{bmatrix}\gamma\\\delta\end{bmatrix}\bar{\vartheta}\begin{bmatrix}\gamma+h\\\delta+g\end{bmatrix}}{\eta^2\bar{\eta}^2}, \tag{78}$$

with the untwisted theta functions corresponding to the untwisted fermions ω^i , while the twisted ones are associated with y^i and similarly for the right-movers. Using Jacobi’s triple product identity $\vartheta_2\vartheta_3\vartheta_4=2\eta^3$, it is straightforward to check that the free fermionic partition function on the r.h.s. of (78) exactly reproduces the bosonic twisted lattice (77) in the orbifold sector (h,g) . Indeed, the triple product identity may be rewritten in the following suggestive form:

其中非扭曲 θ 函数对应非扭曲费米子 ω^i ，扭曲 θ 函数则与 y^i 关联，右动者部分同理。利用雅可比三重积恒等式 $\vartheta_2\vartheta_3\vartheta_4=2\eta^3$ ，可以轻易验证 (78) 右侧的自由费米子配分函数完全重现了轨形扇区 (h,g) 中的玻色扭曲格 (77)。事实上，三重积恒等式可以改写为如下具有启发性的形式:

$$\left|\vartheta\left[\begin{array}{c}\gamma\\ \delta\end{array}\right]\vartheta\left[\begin{array}{c}\gamma+h\\ \delta+g\end{array}\right]\vartheta\left[\begin{array}{c}1-h\\ 1-g\end{array}\right]\right|^2=|2\eta^3|^2, \tag{79}$$

valid whenever the l.h.s. is nonvanishing. In the $(h,g)\neq(0,0)$ sector, out of the four possible values of γ,δ , the l.h.s. of (79) vanishes in exactly two distinct cases, i.e., when $(\gamma,\delta)=(1,1)$ or $(\gamma,\delta)=(1-h,1-g)$. Therefore, for $(h,g)\neq(0,0)$,

只要左侧非零该式就成立。在 $(h,g)\neq(0,0)$ 扇区中，γ,δ 的四个可能取值里，(79) 的左侧恰好会在两种不同情况下为零，即当 $(\gamma,\delta)=(1,1)$ 或 $(\gamma,\delta)=(1-h,1-g)$ 时。因此，对于 $(h,g)\neq(0,0)$ ，

$$\frac{1}{2}\sum_{\gamma,\delta=0,1}\left|\vartheta\left[\begin{array}{c}\gamma\\ \delta\end{array}\right]\vartheta\left[\begin{array}{c}\gamma+h\\ \delta+g\end{array}\right]\right|^2=\left|\frac{2\eta^3}{\vartheta\left[\begin{array}{c}1-h\\ 1-g\end{array}\right]}\right|^2, \tag{80}$$

which indeed reproduces (77) for $(h,g)\neq(0,0)$. In a similar fashion, it is possible to establish relations analogous to Eq. (78) for more complicated orbifold actions, for instance, involving several $\mathbb{Z}_2$ factors or combinations of rotations and translations.

它确实重现了 $(h,g)\neq(0,0)$ 对应的 (77) 式。同理，对于更复杂的轨形作用，例如涉及多个 $\mathbb{Z}_2$ 因子或旋转与平移组合的情况，也可以建立类似 (78) 的关系。

To be concrete, we shall present here the explicit map between the FFF and orbifold formulations of a specific three-generation SO (10) model based on the symmetric $T^6/\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold model and gauge group $\mathrm{SO}(10)\times\mathrm{U}(1)^3\times\mathrm{SO}(8)^2$, which can be constructed using the symmetric basis of Eq. (66). The GGSO matrix of the model, exponentiated in terms of a $\mathbb{Z}_2$ -valued matrix $\mathbf{G}$, is given by

为具体说明，我们将在这里给出基于对称 $T^6/\mathbb{Z}_2\times\mathbb{Z}_2$ 轨道形模型和规范群 $\mathrm{SO}(10)\times\mathrm{U}(1)^3\times\mathrm{SO}(8)^2$ 的特定三代 SO (10) 模型，在 FFF 与轨道形表述之间的显式映射，该模型可利用式 (66) 的对称基构造。该模型的 GGSO 矩阵以 $\mathbb{Z}_2$ 值矩阵 $\mathbf{G}$ 为指数形式，给出如下

$$C\left[\begin{array}{c}\beta_i\\ \beta_j\end{array}\right]=(-1)^{G_{ij}}\ ,\mathbf{G}=\begin{pmatrix}1&1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&0&0&0&1&1&0&0&0&0&0&0\\ 1&1&0&0&0&0&1&1&0&0&0&0&0\\ 1&1&0&1&0&0&0&0&0&0&0&1&0\\ 1&1&1&0&0&0&0&0&0&0&1&0&0\\ 1&1&1&1&0&0&0&1&0&0&0&1&0\\ 1&0&1&0&0&0&1&0&0&0&1&0&0\\ 1&0&0&0&0&0&0&0&1&0&0&0&1\\ 1&1&0&0&0&1&0&1&0&0&1&1&0\\ 1&1&0&0&0&0&0&0&0&0&1&0&0\end{pmatrix}. \tag{81}$$

This particular vacuum appears in Sect. 3.2 of [77]. We will map this model to the orbifold framework by explicitly comparing the one-loop partition functions in both representations. This method is based on

[83, 98], while earlier implementations include [96, 97, 101] . We will do this by organizing the boundary condition vectors $\{\beta_i\}$ into a form convenient for the orbifold representation. An inspection of the basis (66) clearly identifies the $\mathbb{Z}_2 \times \mathbb{Z}_2$ factors with the vectors β_9 and β_{10} , respectively. We, hence, introduce the notation (h_1, g_1) and (h_2, g_2) to refer to the orbifold parameters labeling the twisted sectors and projection parameters of the $\mathbb{Z}_2$ factors generated by β_{10} and β_9 , respectively. We now group together fermions sharing the same boundary conditions and assign the following boundary condition labels to the fermions appearing in each one of the vectors:

> 该特殊真空出现在文献 [77] 的 3.2 节。我们通过显式比较两种表示下的单圈配分函数，将该模型映射到轨道形框架。该方法基于文献 [83, 98]，早期实现工作包括 [96, 97, 101] 。我们通过将边界条件向量 $\{\beta_i\}$ 整理为适合轨道形表示的形式完成这项工作。对基 (66) 的检验可清晰将 $\mathbb{Z}_2 \times \mathbb{Z}_2$ 因子分别对应到向量 β_9 和 β_{10} 。因此，我们引入记号 (h_1, g_1) 和 (h_2, g_2) ，分别表示标记扭区的轨道形参数，以及分别由 β_{10} 和 β_9 生成的 $\mathbb{Z}_2$ 因子的投影参数。现在我们将共享相同边界条件的费米子分组，并给每个向量中出现的费米子赋予以下边界条件标签:

$$\beta_2 \to (a, b)$$

$$\beta_{2+i} \to (\gamma_i, \delta_i)\,, (i = 1, 2, \ldots 6)\,,$$

$$\beta_1 + \beta_2 + \sum_{i=1}^{6} \beta_{2+i} + \beta_{11} + \beta_{12} \to (k, \ell)\,, \tag{82}$$

$$\beta_{11} + \beta_{12} \to (\rho, \sigma)\,,$$

The logic behind this correspondence is as follows. Before any twist is introduced, the (common) boundary conditions of RNS fermions are denoted (a, b) , those of auxiliary fermions realizing the lattice are (γ_i, δ_i) , those of fermions realizing the first E_8 factor are (k, ℓ) , and those realizing the second E_8 are (ρ, σ) .

> 该对应的逻辑如下。在引入任何扭转之前，RNS 费米子的 (公共) 边界条件记为 (a, b) ，实现晶格的辅助费米子的边界条件记为 (γ_i, δ_i) ，实现第一个 E_8 因子的费米子边界条件记为 (k, ℓ) ，实现第二个 E_8 因子的费米子边界条件记为 (ρ, σ) 。

The orbifold twists act on top of these assignments and alter the boundary conditions of the fermions associated with β_9, β_{10} and β_{12} . Specifically, the twist in the boundary conditions of the $\mathbb{Z}_2$ orbifold factor associated with β_9 is labeled by (h_2, g_2) , while the corresponding twist due to the $\mathbb{Z}_2$ associated with β_{10} is labeled by (h_1, g_1) . Finally, (H, G) labels the additional orbifold needed to break the second E_8 down to a product of SO (8) factors. In terms of our assignment, the fermions in β_9, β_{10} and β_{12} receive a twist of their boundary conditions as

> 轨道形扭转作用在上述赋值之上，会改变与 β_9, β_{10} 和 β_{12} 关联的费米子的边界条件。具体而言，与 β_9 关联的 $\mathbb{Z}_2$ 轨道形因子的边界条件扭转标记为 (h_2, g_2) ，而与 β_{10} 关联的 $\mathbb{Z}_2$ 对应的扭转标记为 (h_1, g_1) 。最后，(H, G) 标记将第二个 E_8 破缺为 SO (8) 因子乘积所需的额外轨道形。根据我们的赋值，β_9, β_{10} 和 β_{12} 中的费米子的边界条件获得的扭转如下

$$\left.\begin{array}{l}(h_2, g_2)\\(h_1, g_1)\\(H, G)\end{array}\right\}\text{ twists b.c. of fermions in }\begin{cases}\beta_9\\\beta_{10}\\\beta_{12}\end{cases}\tag{83}$$

To construct the orbifold partition function, it is instructive to start with the $E_8 \times\ E_8$ heterotic string, compactified on a T^6 at the factorized point $T^6 = \left(S^1\right)^6$ where all radii are equal to $\sqrt{\alpha'/2}$. The partition function of this theory takes the form

> 为构造轨道形配分函数，从 $E_8 \times\ E_8$ 杂化弦出发十分直观，它在因子化点 $T^6 = \left(S^1\right)^6$ 紧致化于 T^6 ，该点所有半径都等于 $\sqrt{\alpha'/2}$ 。该理论的配分函数形式为

$$Z = \frac{1}{\eta^{12}\bar{\eta}^{24}}\left[\frac{1}{2}\sum_{a,b=0,1}(-1)^{a+b+ab}\vartheta^4\left[\begin{array}{c}a\\b\end{array}\right]\right]\tag{84}$$

$$\times\Gamma_{6,6}\left[\frac{1}{2}\sum_{k,\ell=0,1}\bar{\vartheta}^8\left[\begin{array}{c}k\\\ell\end{array}\right]\frac{1}{2}\sum_{\rho,\sigma=0,1}\bar{\vartheta}^8\left[\begin{array}{c}\rho\\\sigma\end{array}\right]\right].$$

In this notation, the RNS fermions $\left\{\psi^\mu, \chi^{1\cdots 6}\right\}$ are given the same boundary conditions (a, b) along the two cycles of the worldsheet torus, and these are summed over all spin structures $a, b = 0, 1$ with the correct spin-statistics (GSO) phase $(-1)^{a+b+ab}$. Note that the additional term ab in the phase is simply a GSO convention. The right-moving Kač-Moody fermions $\left\{\bar{\psi}^{1\ldots 5}, \bar{\eta}^1, \bar{\eta}^2, \bar{\eta}^3\right\}$ realize the first E_8 factor and are similarly given the same boundary conditions (k, ℓ) , which after appropriate summation over $k, \ell = 0, 1$ build up the anti-chiral E_8 lattice. Similarly, $\left\{\bar{\phi}^{1\ldots 8}\right\}$ realize the second E_8 factor and are assigned the boundary conditions (ρ, σ) . The Narain lattice of signature $(6, 6)$ corresponding to the fermionic point is simply a product of six $(1, 1)$ lattices of the type given in Eq. (17), each one realized by the auxiliary fermions $\left\{y^i, \omega^i, \bar{y}^i, \bar{\omega}^i\right\}$:

> 在此记号下，RNS 费米子 $\{\psi^\mu, \chi^{1\cdots 6}\}$ 在世界面轮胎的两个闭链上被赋予相同的边界条件 (a, b) ，我们对所有满足正确自旋统计 (GSO) 相位 $(-1)^{a+b+ab}$ 的自旋结构 $a, b = 0, 1$ 求和。注意相位中的附加项 ab 仅是一种 GSO 约定。右行卡茨-穆迪费米子 $\left\{\bar{\psi}^{1\ldots 5}, \bar{\eta}^1, \bar{\eta}^2, \bar{\eta}^3\right\}$ 实现了第一个 E_8 因子，同样被赋予相同的边界条件 (k, ℓ) ，对 $k, \ell = 0, 1$ 完成恰当求和后，就构造出反手性 E_8 格。类似地，$\left\{\bar{\phi}^{1\ldots 8}\right\}$ 实现第二个 E_8 因子，被赋予边界条件 (ρ, σ) 。对应费米点的符号差为 $(6, 6)$ 的纳莱恩格，恰好是六个式 (17) 所示类型的 $(1, 1)$ 格的直积，每个格都由辅助费米子 $\left\{y^i, \omega^i, \bar{y}^i, \bar{\omega}^i\right\}$ 实现:

$$\Gamma_{6,6} = \prod_{i=1}^{6}\frac{1}{2}\sum_{\gamma_i,\delta_i=0,1}\vartheta\begin{bmatrix}\gamma_i\\\delta_i\end{bmatrix}\bar{\vartheta}\begin{bmatrix}\gamma_i\\\delta_i\end{bmatrix}.\tag{85}$$

This is precisely the theory generated by the basis vectors $\beta_1, \beta_2, \ldots, \beta_8$ together with the combination $\beta_{11} + \beta_{12}$ (the latter simply breaks SO (32) to the product of E8's).

> 这正是由基矢 $\beta_1, \beta_2, \ldots, \beta_8$ 加上组合 $\beta_{11} + \beta_{12}$ 生成的理论 (后者仅将 SO (32) 破缺为多个 E8 的直积)。

Next we consider the orbifold factors. First, notice that including β_{12} in our basis [14] has the effect of splitting up the boundary conditions of $\bar{\phi}^{1\ldots 4}$ and $\bar{\phi}^{5\ldots 8}$ and, hence, may break the second E_8 down to SO (8)×SO (8)

. In the orbifold representation, we can implement this by defining a $\mathbb{Z}_2$ twist of the boundary conditions of $\overline{\phi}^{5\ldots8}$, labeled by (H,G), and by replacing

接下来我们考虑轨形因子。首先注意，在基 [14] 中加入 β_{12} 会拆分 $\overline{\phi}^{1\ldots4}$ 和 $\overline{\phi}^{5\ldots8}$ 的边界条件，因此可能将第二个 E_8 破缺为 $SO(8)\times SO(8)$。在轨形表示中，我们可以通过如下方式实现这一点: 定义 $\overline{\phi}^{5\ldots8}$ 边界条件的 $\mathbb{Z}_2$ 扭，由 (H,G) 标记，然后替换

$$\overline{\vartheta}^{8}\begin{bmatrix}\rho\\ \sigma\end{bmatrix}\to(-1)^{HG}\overline{\vartheta}^{4}\begin{bmatrix}\rho\\ \sigma\end{bmatrix}\overline{\vartheta}^{4}\begin{bmatrix}\rho+H\\ \sigma+G\end{bmatrix}, \tag{86}$$

where the phase $(-1)^{HG}$ is required by modularity. [15] Similarly, incorporating the $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold action associated with the elements β_9 and β_{10} results in the replacements

其中相位 $(-1)^{HG}$ 是模不变性的要求。[15] 类似地，引入对应元素 β_9 和 β_{10} 的 $\mathbb{Z}_2\times\mathbb{Z}_2$ 轨形作用后，得到替换关系

(87)

$$\vartheta^{4}\begin{bmatrix}a\\ b\end{bmatrix}\to\vartheta\begin{bmatrix}a\\ b\end{bmatrix}\vartheta\begin{bmatrix}a+h_1\\ b+g_1\end{bmatrix}\vartheta\begin{bmatrix}a+h_2\\ b+g_2\end{bmatrix}\vartheta\begin{bmatrix}a-h_1-h_2\\ b-g_1-g_2\end{bmatrix},$$
$$\overline{\vartheta}^{8}\begin{bmatrix}k\\ \ell\end{bmatrix}\to\overline{\vartheta}^{5}\begin{bmatrix}k\\ \ell\end{bmatrix}\overline{\vartheta}\begin{bmatrix}k+h_1\\ \ell+g_1\end{bmatrix}\overline{\vartheta}\begin{bmatrix}k+h_2\\ \ell+g_2\end{bmatrix}\overline{\vartheta}\begin{bmatrix}k-h_1-h_2\\ \ell-g_1-g_2\end{bmatrix},$$

for the left- and right-moving fermions in the RNS and Kač-Moody sectors, respectively, while the lattice fermions are replaced by

分别对应 RNS 区和卡茨-穆迪区的左行与右行费米子，而格费米子则被替换为

$$\left|\vartheta\begin{bmatrix}\gamma_1\\ \delta_1\end{bmatrix}\vartheta\begin{bmatrix}\gamma_2\\ \delta_2\end{bmatrix}\right|^2\to\left|\vartheta\begin{bmatrix}\gamma_1\\ \delta_1\end{bmatrix}\vartheta\begin{bmatrix}\gamma_1+h_1\\ \delta_1+g_1\end{bmatrix}\vartheta\begin{bmatrix}\gamma_2\\ \delta_2\end{bmatrix}\vartheta\begin{bmatrix}\gamma_2+h_1\\ \delta_2+g_1\end{bmatrix}\right|,$$
$$\left|\vartheta\begin{bmatrix}\gamma_3\\ \delta_3\end{bmatrix}\vartheta\begin{bmatrix}\gamma_4\\ \delta_4\end{bmatrix}\right|^2\to\left|\vartheta\begin{bmatrix}\gamma_3\\ \delta_3\end{bmatrix}\vartheta\begin{bmatrix}\gamma_3+h_2\\ \delta_3+g_2\end{bmatrix}\vartheta\begin{bmatrix}\gamma_4\\ \delta_4\end{bmatrix}\vartheta\begin{bmatrix}\gamma_4+h_2\\ \delta_4+g_2\end{bmatrix}\right|,$$
$$\left|\vartheta\begin{bmatrix}\gamma_5\\ \delta_5\end{bmatrix}\vartheta\begin{bmatrix}\gamma_6\\ \delta_6\end{bmatrix}\right|^2\to\left|\vartheta\begin{bmatrix}\gamma_5\\ \delta_5\end{bmatrix}\vartheta\begin{bmatrix}\gamma_5-h_1-h_2\\ \delta_5-g_1-g_2\end{bmatrix}\vartheta\begin{bmatrix}\gamma_6\\ \delta_6\end{bmatrix}\vartheta\begin{bmatrix}\gamma_6-h_1-h_2\\ \delta_6-g_1-g_2\end{bmatrix}\right|.$$

(88)

In terms of boundary condition assignments, factors of the form $|\vartheta|$ should be seen as a short-hand of the real fermion contribution $\vartheta^{1/2}\overline{\vartheta}^{1/2}$. The partition function can therefore be cast into the form

从边界条件赋值的角度看，形如 $|\vartheta|$ 的因子可看作实费米子贡献 $\vartheta^{1/2}\overline{\vartheta}^{1/2}$ 的简写。因此配分函数可以写成如下形式

$$
\begin{aligned}
Z = \frac{1}{\eta^{12}\overline{\eta}^{24}} \frac{1}{2^{12}} \sum_{\{A\},\{B\}} (-1)^{a+b+ab+HG+\Phi\left[\begin{smallmatrix}\{A\}\\ \{B\}\end{smallmatrix}\right]} \\
\times \vartheta\begin{bmatrix} a \\ b \end{bmatrix} \vartheta\begin{bmatrix} a+h_1 \\ b+g_1 \end{bmatrix} \vartheta\begin{bmatrix} a+h_2 \\ b+g_2 \end{bmatrix} \vartheta\begin{bmatrix} a-h_1-h_2 \\ b-g_1-g_2 \end{bmatrix} \\
\times \left| \vartheta\begin{bmatrix} \gamma_1 \\ \delta_1 \end{bmatrix} \vartheta\begin{bmatrix} \gamma_1+h_1 \\ \delta_1+g_1 \end{bmatrix} \vartheta\begin{bmatrix} \gamma_2 \\ \delta_2 \end{bmatrix} \vartheta\begin{bmatrix} \gamma_2+h_1 \\ \delta_2+g_1 \end{bmatrix} \right| \\
\times \left| \vartheta\begin{bmatrix} \gamma_3 \\ \delta_3 \end{bmatrix} \vartheta\begin{bmatrix} \gamma_3+h_2 \\ \delta_3+g_2 \end{bmatrix} \vartheta\begin{bmatrix} \gamma_4 \\ \delta_4 \end{bmatrix} \vartheta\begin{bmatrix} \gamma_4+h_2 \\ \delta_4+g_2 \end{bmatrix} \right| \\
\times \left| \vartheta\begin{bmatrix} \gamma_5 \\ \delta_5 \end{bmatrix} \vartheta\begin{bmatrix} \gamma_5-h_1-h_2 \\ \delta_5-g_1-g_2 \end{bmatrix} \vartheta\begin{bmatrix} \gamma_6 \\ \delta_6 \end{bmatrix} \vartheta\begin{bmatrix} \gamma_6-h_1-h_2 \\ \delta_6-g_1-g_2 \end{bmatrix} \right| \\
\times \overline{\vartheta}^5\begin{bmatrix} k \\ \ell \end{bmatrix} \overline{\vartheta}\begin{bmatrix} k+h_1 \\ \ell+g_1 \end{bmatrix} \overline{\vartheta}\begin{bmatrix} k+h_2 \\ \ell+g_2 \end{bmatrix} \overline{\vartheta}\begin{bmatrix} k-h_1-h_2 \\ \ell-g_1-g_2 \end{bmatrix} \\
\times \overline{\vartheta}^4\begin{bmatrix} \rho \\ \sigma \end{bmatrix} \overline{\vartheta}^4\begin{bmatrix} \rho+H \\ \sigma+G \end{bmatrix},
\end{aligned} \tag{89}
$$

---

[14] Or, equivalently, having β_{11} and β_{12} as separate elements of the basis

> [14] 或者等价地，将 β_{11} 和 β_{12} 作为基中两个独立元素

[15] Note that if (H, G) does not couple to the rest of the partition function, summing over $H, G = 0{,}1$ in the r.h.s. of (86) reproduces the l.h.s. and E_8 is regenerated (as it should).

> [15] 注意到，若 (H, G) 不与配分函数的其余部分耦合，对 (86) 右侧的 $H, G = 0{,}1$ 求和即可重现左侧，且会重新生成 E_8 (这是理应发生的)。

---

where $\{A\} = \{a, k, \rho, \{\gamma_i\}, h_1, h_2, H\}$ and $\{B\} = \{b, \ell, \sigma, \{\delta_i\}, g_1, g_2, G\}$ collectively denote the summation parameters of upper and lower arguments of the theta functions, respectively, while Φ represents an as-yet unspecified phase that may a priori depend on all summation variables in $\{A\}$ and $\{B\}$. Our next step is to constrain this dependence and show how it can be obtained from the GGSO coefficients (81). In what follows, it will be convenient to assemble the summation parameters into the 12-dimensional column vectors:

> 其中 $\{A\} = \{a, k, \rho, \{\gamma_i\}, h_1, h_2, H\}$ 和 $\{B\} = \{b, \ell, \sigma, \{\delta_i\}, g_1, g_2, G\}$ 分别共同表示 theta 函数上、下自变量的求和参数，而 Φ 表示一个尚未确定的相位，该相位原则上可依赖于 $\{A\}$ 和 $\{B\}$ 中的所有求和变量。我们下一步将约束这种依赖关系，并说明如何从 GGSO 系数 (81) 得到它。在下文中，我们方便将求和参数组合为 12 维列向量:

$$
\mathbf{A}^T = (a, k, \rho, \gamma_1, \dots, \gamma_6, h_1, h_2, H)\,, \mathbf{B}^T = (b, \ell, \sigma, \delta_1, \dots, \delta_6, g_1, g_2, G)\,.
$$

(90)

It is straightforward to see that Φ must be invariant under the action of modular transformations on the characteristics. Indeed, using the modular transformation properties of Jacobi theta functions, it can be explicitly checked that the integrand Z/τ_2 of one-loop vacuum amplitude is modular invariant [16] provided Φ is left invariant (modulo 2) under the action of the modular group on the space of theta characteristics $\{A\}, \{B\}$. Specifically, for the $\tau \to \tau + 1$ transformation, the relevant action is

不难看出，Φ 在特征量的模变换作用下必须不变。实际上，利用雅可比 theta 函数的模变换性质可以明确验证: 只要 Φ 在模群作用于 theta 特征空间 $\{A\}, \{B\}$ 时保持不变 (模 2 意义下)，单圈真空振幅的被积函数 Z/τ_2 就是模不变的 [16] 。具体而言，对 $\tau \to \tau + 1$ 变换，相关作用为

$$\begin{pmatrix} b \\ \ell \\ \sigma \\ \delta_i \end{pmatrix} \to \begin{pmatrix} b+a-1 \\ \ell+k-1 \\ \sigma+\rho-1 \\ \delta_i+\gamma_i-1 \end{pmatrix} \text{and} \begin{pmatrix} g_1 \\ g_2 \\ G \end{pmatrix} \to \begin{pmatrix} g_1+h_1 \\ g_2+h_2 \\ G+H \end{pmatrix}, \tag{91}$$

---

[16] Note that the partition function Z defined as in (89) does not include the nonanalytic contribution $\left(\sqrt{\tau_2}\right)^2 = \tau_2^{-1}$, associated with the two transverse non-compact coordinates of $4d$ space-time. As a result, the modular invariance conditions should actually be imposed on Z/τ_2 .

[16] 注意，按 (89) 定义的配分函数 Z 不包含非解析贡献项 $\left(\sqrt{\tau_2}\right)^2 = \tau_2^{-1}$ ，该贡献与 $4d$ 时空的两个横向非紧致坐标相关。因此，模不变性条件实际上应当施加于 Z/τ_2 。

---

while for the $\tau \to -1/\tau$ transformation,

而对 $\tau \to -1/\tau$ 变换,

$$\begin{pmatrix} a \\ k \\ \rho \\ \gamma_i \end{pmatrix} \leftrightarrow \begin{pmatrix} b \\ \ell \\ \sigma \\ \delta_i \end{pmatrix} \text{and} \begin{pmatrix} h_1 \\ h_2 \\ H \end{pmatrix} \leftrightarrow \begin{pmatrix} g_1 \\ g_2 \\ G \end{pmatrix}. \tag{92}$$

The exact determination of Φ specifies the orbifold model and is necessary in order to complete the map from the FFF. Indeed, a comparison of Eq. (89) with the general form Eq. (49) of the partition function in the FFF (stripped off the τ_2 pre-factors and the modular integral) indicates that the freedom in consistently picking the GGSO coefficients corresponds to the freedom in (consistently) choosing the modular invariant phase Φ .

Φ 的精确确定明确了轨形模型，也是完成从 FFF 出发的映射所必需的。事实上，将 (89) 式与 FFF 中配分函数的一般形式 (49) 式 (去除 τ_2 前置因子和模积分后) 对比可知，一致选取 GGSO 系数的自由度对应于 (一致地) 选取模不变相位 Φ 的自由度。

To be precise, arrange all theta function factors in (89) in the same order that the corresponding fermions enter the basis element β_1 (using complex fermions whenever possible) and define **a** and **b** to be the vectors of their upper and lower characteristics, respectively:

准确来说，将 (89) 中所有 theta 函数因子按对应费米子进入基元 β_1 的相同顺序排列 (尽可能使用复费米子)，再分别定义 **a** 和 **b** 为它们上特征和下特征构成的向量:

$$\mathbf{a} = (a, a + h_1, a + h_2, a - h_1 - h_2, \dots), \mathbf{b} = (b, b + g_1, b + g_2, b - g_1 - g_2, \dots).$$

(93)

For simplicity, in the above formula, we only explicitly display the characteristics of the first few left-moving fermions $\psi^\mu, \chi^{1,2}, \chi^{3,4}$, and $\chi^{5,6}$, but it is clear how to complete the process for all fermions in β_1 , including also the right-moving ones. The end result looks very similar to the boundary condition vectors $\alpha, \beta \in \Xi$ in Eq. (47) of the FFF, except for the fact that the elements of α, β actually correspond to the reduced representatives in the interval $(-1, 1]$, i.e., for any fermion f , they satisfy [17]$\alpha(f), \beta(f) \in (-1, 1]$. On the other hand, the elements of $\mathbf{a}, \mathbf{b}$ in Eq. (93) do not necessarily lie in $(-1, 1]$, but may always brought into this interval by adding suitable even integers. To this end, we denote by [a] the reduced representative of **a** such that all its elements are in $(-1, 1]$. Now, observe that the set $\{[\mathbf{a}]\}$ of (reduced) boundary condition vectors obtained by allowing the 12 summation parameters $a, k, \rho, \gamma_i, h_1, h_2, H \in \{0, 1\}$ to span all allowed values is isomorphic to the subgroup Ξ generated by the 12 basis vectors $\{\beta_i\}$ in the FFF and similarly for $\{[\mathbf{b}]\}$. Therefore, there exists a bijective map between the GGSO coefficients of the FFF and the phase $\Phi\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}$ of the representation (89).

为简化起见，在上述公式中，我们仅明确展示了前几个左行费米子 $\psi^\mu, \chi^{1,2}, \chi^{3,4}$ 和 $\chi^{5,6}$ 的特征，但对于 β_1 中包含右行费米子在内的所有费米子，完成该过程的方式十分清晰。最终结果与 FFF 第 (47) 式中的边界条件向量 $\alpha, \beta \in \Xi$ 非常相似，区别仅在于 α, β 的元素实际上对应区间 $(-1, 1]$ 中的约化代表元，即对任意费米子 f，它们满足 [17]$\alpha(f), \beta(f) \in (-1, 1]$。另一方面，第 (93) 式中 $\mathbf{a}, \mathbf{b}$ 的元素不一定落在 $(-1, 1]$ 内，但总可以通过加上合适的偶数将其归入该区间。为此，我们用 [a] 表示 **a** 的约化代表元，其所有元素都位于 $(-1, 1]$ 中。现在可以看到，令 12 个求和参数 $a, k, \rho, \gamma_i, h_1, h_2, H \in \{0, 1\}$ 取遍所有允许值后得到的 (约化) 边界条件向量集合 $\{[\mathbf{a}]\}$，同构于 FFF 中由 12 个基向量 $\{\beta_i\}$ 生成的子群 Ξ，$\{[\mathbf{b}]\}$ 的情况也类似。因此，FFF 的 GGSO 系数和表示 (89) 的相位 $\Phi\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}$ 之间存在一个双射。

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[17] Of course, the particular example we consider here involves only real fermions, which can be either periodic or antiperiodic so that $\alpha(f), \beta(f) \in \{0, 1\}$. However, the method we present here is general and straightforwardly generalizes to rational values as well.

[17] 当然，我们此处讨论的具体例子仅涉及实费米子，它只能是周期性或反周期性的，因此 $\alpha(f), \beta(f) \in \{0, 1\}$。但本文提出的方法具有普适性，可直接推广至有理数值的情况。

In order to determine Φ in terms of the GSSO coefficients, we express the partition function (89) in the form

为了用 GSSO 系数确定 Φ，我们将配分函数 (89) 写为如下形式

$$\tau_2\eta^2\overline{\eta}^2 Z = \frac{1}{2^{12}}\sum_{\mathbf{a},\mathbf{b}} F\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} Z\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} \tag{94}$$

where

其中

$$F\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} = (-1)^{a+b+ab+HG+\Phi\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}}, \tag{95}$$

contains the phase factor, while $Z\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}$ simply contains all left- and right-moving theta function factors (divided by corresponding Dedekind eta functions) as in (33), arranged in the same order that the fermion boundary conditions appear in $\mathbf{a}$ and b. In our example, the additive group Ξ is a direct sum of $12\mathbb{Z}_2$ factors and, hence, $|\Xi| = 2^{12}$. The form (94) is very similar to (49), except for two important differences. Firstly,(49) uses the Θ -convention (34) for the theta functions instead of the ϑ one and, secondly, the boundary condition vectors α, β in (49) are of the reduced type (by construction), whereas the $\mathbf{a}, \mathbf{b}$ defined by (93) are not. The first discrepancy is easy to take into account, in view of (34), and amounts to the replacement $\theta \to \Theta$ with the simultaneous inclusion of a phase $\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\mathbf{a}\cdot\mathbf{b}}$. The second discrepancy can be accounted for by noting the identity

包含相位因子，而 $Z\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}$ 仅包含所有左行和右行 θ 函数因子 (除以对应的戴德金 η 函数)，如 (33) 所示，按照费米子边界条件在 $\mathbf{a}$ 和 b 中出现的顺序排列。在我们的例子中，加法群 Ξ 是 $12\mathbb{Z}_2$ 因子的直和，因此 $|\Xi| = 2^{12}$ 。形式 (94) 与 (49) 非常相似，除了两处重要区别。第一，(49) 对 θ 函数采用 Θ 约定而非 ϑ 约定，第二，(49) 中的边界条件向量 α, β (按构造) 是约化类型的，而由 (93) 定义的 $\mathbf{a}, \mathbf{b}$ 不是。根据 (34)，第一处差异很容易处理，只需替换 $\theta \to \Theta$ 并同时引入相位 $\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\mathbf{a}\cdot\mathbf{b}}$ 即可。第二处差异可以通过如下恒等式处理:

$$\Theta\begin{bmatrix}a\\ b\end{bmatrix}(z;\tau) = \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}(a-[a])b}\Theta\begin{bmatrix}[a]\\ [b]\end{bmatrix}(z;\tau), \tag{96}$$

which relates the Θ ’s with unreduced characteristics to those of reduced type. In particular, the reduced Θ ’s on the r.h.s. are now periodic under both upper and lower arguments. Taking both points into account, (94) can be finally expressed in the form

它将具有未约化特征标得 Θ 与约化类型的 Θ 联系起来。特别地，右侧的约化 Θ 现在对上下自变量都具有周期性。综合这两点，式 (94) 最终可写为

$$\tau_2\eta^2\overline{\eta}^2 Z = \frac{1}{2^{12}}\sum_{\mathbf{a},\mathbf{b}} C\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} \hat{Z}\begin{bmatrix}[\mathbf{a}]\\ [\mathbf{b}]\end{bmatrix} \tag{97}$$

with

其中

$$C\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}=F\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\mathbf{a}\cdot\mathbf{b}-\frac{\mathrm{i}\pi}{2}(\mathbf{a}-[\mathbf{a}])\cdot[\mathbf{b}]}. \tag{98}$$

Direct comparison of (97) with (49) now allows for an exact term-by-term match of the coefficients, once the appropriate change of basis from the $\alpha,\beta\in\Xi$ to the $\mathbf{A},\mathbf{B}$ is established. In other words, the set of $C\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}$ ' s in the above equation is precisely isomorphic to the set of GGSO coefficients $C\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ of the FFF.

只要建立了从 $\alpha,\beta\in\Xi$ 到 $\mathbf{A},\mathbf{B}$ 的适当基变换，将式 (97) 与式 (49) 直接对比，就可以得到系数逐项的精确匹配。换言之，上述方程中的 $C\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}$ 集合与 FFF 的 GGSO 系数 $C\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ 集合恰好同构。

Furthermore, plugging (98) into the modular invariance conditions (38), (39), and (46), we extract the corresponding conditions on the F ' s that correspond to our basis:

此外，将式 (98) 代入模不变性条件 (38)、(39) 和 (46)，我们可以提取出对应于我们基的 F 所满足的相应条件:

$$\begin{gathered}F\begin{bmatrix}\mathbf{a}\\ \mathbf{b}-\mathbf{a}+1\end{bmatrix}=(-1)^{1+a^2+H^2}F\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix},\\ F\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}=F\begin{bmatrix}\mathbf{b}\\ -\mathbf{a}\end{bmatrix}\\ F\begin{bmatrix}\mathbf{a}\\ \mathbf{b}+\mathbf{b}'\end{bmatrix}=(-1)^{a}F\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}F\begin{bmatrix}\mathbf{a}\\ \mathbf{b}'\end{bmatrix}.\end{gathered} \tag{99}$$

In particular, notice that, although the GGSO phases C are not symmetric under exchange of upper and lower characteristics, this asymmetry is precisely balanced by the asymmetric phase on the r.h.s of (98), such that the F ' s turn out to be symmetric. Since Ξ is a direct sum of $\mathbb{Z}_2$ ' s, the phases are necessarily real. It is then easy to see that the corresponding conditions on the exponent $\Phi\begin{bmatrix}a\\ b\end{bmatrix}$ of the phase appearing in (89) read

特别注意，尽管 GGSO 相位 C 在上下特征标交换下不具有对称性，但这种不对称性恰好被式 (98) 右侧的不对称相位抵消，因此得到的 F 是对称的。由于 Ξ 是 $\mathbb{Z}_2$ 的直和，相位必然为实。不难看出，式 (89) 中出现的相位指数 $\Phi\begin{bmatrix}a\\ b\end{bmatrix}$ 所满足的对应条件为

$$\Phi\begin{bmatrix}\mathbf{a}\\ \mathbf{b}-\mathbf{a}+1\end{bmatrix}=\Phi\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}(\bmod 2)$$

$$\Phi\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix} = \Phi\begin{bmatrix}\mathbf{b}\\-\mathbf{a}\end{bmatrix} (\bmod\ 2) \tag{100}$$

$$\Phi\begin{bmatrix}\mathbf{a}\\\mathbf{b}+\mathbf{b}'\end{bmatrix} = \Phi\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix} + \Phi\begin{bmatrix}\mathbf{a}\\\mathbf{b}'\end{bmatrix} (\bmod\ 2).$$

The first two are the one-loop conditions derived earlier and simply express the fact that $(-1)^{\Phi}$ is modular invariant, i.e., under (91) and (92). The third one is the two-loop condition arising from factorization and implies that Φ can be at most linear in the lower characteristics. Together with the second condition, which requires symmetry under the exchange $\mathbf{a} \leftrightarrow \mathbf{b}$, the phase Φ is restricted to be of the form

前两个是此前推导得到的单圈条件，它们仅说明了 $(-1)^{\Phi}$ 在式 (91) 和 (92) 下是模不变的。第三个是来自因子化的两圈条件，它表明 Φ 对下特征标至多是线性的。结合要求 $\mathbf{a} \leftrightarrow \mathbf{b}$ 交换下对称性的第二个条件，相位 Φ 被限制为如下形式

$$\Phi\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix} = \mathbf{A}^T\mathbf{M}\mathbf{B} = \sum_{i,j=1}^{12} A_i M_{ij} B_j, \tag{101}$$

where $\mathbf{M}$ is a constant 12×12 symmetric matrix (defined modulo 2), which is in one-to-one correspondence with the GGSO matrix (81). The matrix $\mathbf{M}$ is further constrained by the first condition in (100), which implies $M_{ii} = \sum_{j=1}^{9} M_{ij}$ for all $i = 1, \ldots, 12$. However, not all those conditions are independent. Their sum trivially vanishes (modulo 2), since it involves adding the upper and lower triangular parts which, however, are equal due to the symmetry of $\mathbf{M}$. In general, for n basis vectors, there are $\frac{1}{2}n(n-1)$ conditions from the symmetry of M_{ij} but only $(n-1)$ conditions from the modular T -transformation. As a result, M_{ij} has $\frac{1}{2}n(n-1)+1$ independent elements, which is consistent with the fact that there are $2^{\frac{n(n-1)}{2}+1}$ fermionic models with the given basis.

其中 $\mathbf{M}$ 是一个常数 12×12 对称矩阵 (模 2 定义)，它与式 (81) 的 GGSO 矩阵一一对应。矩阵 $\mathbf{M}$ 还受到式 (100) 中第一个条件的约束，该条件意味着对所有 $i = 1, \ldots, 12$ 都有 $M_{ii} = \sum_{j=1}^{9} M_{ij}$ 。但并非所有这些条件都是独立的，它们的和模 2 平凡为零，因为求和涉及上下三角部分，而由于 $\mathbf{M}$ 的对称性，这两部分相等。一般来说，对于 n 个基向量，M_{ij} 的对称性会给出 $\frac{1}{2}n(n-1)$ 个条件，而模 T 变换仅给出 $(n-1)$ 个条件。因此，M_{ij} 有 $\frac{1}{2}n(n-1)+1$ 个独立元，这与给定基下存在 $2^{\frac{n(n-1)}{2}+1}$ 个费米子模型的结论一致。

We will now establish the precise relation between the 12-dimensional matrix of GGSO coefficients $\mathbf{C}\begin{bmatrix}\beta_i\\\beta_j\end{bmatrix}$ and $\mathbf{M}$. Until we have to deal with the asymmetric phase on the r.h.s. of (98), we can work modulo 2 and effectively identify a with [a]. Take an arbitrary element $\alpha = \mathbf{a} \in \Xi$ and expand it in the $\{\beta_i\}$ basis of the FFF as $\mathbf{a} = \lambda_i(\mathbf{a})\beta_i$. The components $\lambda_i(\mathbf{a})$ will clearly be linear combinations of the A_i ’ s, namely, $\lambda_i(\mathbf{a}) = \tilde{S}_{ij}A_j(\mathbf{a})$ for some invertible matrix $\tilde{S}_{ij}$ independent of $\mathbf{a}$ which one may easily identify. We can then interpret $A_i(\mathbf{a})$ as the linear functions

我们现在将建立 GGSO 系数的 12 维矩阵 $\mathbf{C}\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix}$ 与 $\mathbf{M}$ 之间的精确关系。在需要处理式 (98) 右侧的非对称相之前，我们可以在模 2 下运算，并将 a 等价于 [a]。任取元素 $\alpha = \mathbf{a} \in \Xi$，将其在 FFF 的 $\{\beta_i\}$ 基下展开为 $\mathbf{a} = \lambda_i(\mathbf{a})\beta_i$。分量 $\lambda_i(\mathbf{a})$ 显然是 A_i 的线性组合，即 $\lambda_i(\mathbf{a}) = \tilde{S}_{ij}A_j(\mathbf{a})$，其中可逆矩阵 $\tilde{S}_{ij}$ 与 $\mathbf{a}$ 无关，很容易直接确定。我们可以将 $A_i(\mathbf{a})$ 理解为对偶基 $\mathbf{C}\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix}$ 下的线性函数

$$A_i(\mathbf{a}) = \left(\tilde{S}^{-1}\right)_{ij}\beta_j^{\star}(\mathbf{a}), \tag{102}$$

in terms of the dual basis $\left\{\beta_j^{\star}\right\}$. In particular, $A_j(\beta_i) = \left(\tilde{S}^{-T}\right)_{ij} \equiv S_{ij}$. Having determined the matrix encoding the change of basis S_{ij} allows all parameters $a, b, k, \ell, \ldots$ to be uniquely fixed in terms of the basis vectors β_i, β_j appearing as upper and lower characteristics of $C\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix}$. For example, $a \to A_1(\beta_i) = S_{i1}$, while $b \to B_1(\beta_j) = S_{j2}$ and similarly for the others. In this notation, we have

以对偶基 $\left\{\beta_j^{\star}\right\}$ 表示。特别地，$A_j(\beta_i) = \left(\tilde{S}^{-T}\right)_{ij} \equiv S_{ij}$。确定了编码基变换的矩阵 S_{ij} 后，即可利用 $C\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix}$ 的上下特征中出现的基向量 β_i, β_j 唯一确定所有参数 $a, b, k, \ell, \ldots$。例如，$a \to A_1(\beta_i) = S_{i1}$，而 $b \to B_1(\beta_j) = S_{j2}$，其余参数同理可得。采用该记号，我们得到

$$\Phi\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix} = \sum_{r,s=1}^{12} A_r(\beta_i)M_{rs}B_s(\beta_j) = \left(\mathbf{SMS}^T\right)_{ij}. \tag{103}$$

Now define the $\mathbb{Z}_2$-valued GGSO exponent matrix $G_{ij} \in \{0,1\}$ via $C\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix} = (-1)^{G_{ij}}$ and assemble the exponents (modulo 2) of the phase factors of (95) and (98) into

现在通过 $C\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix} = (-1)^{G_{ij}}$ 定义取值于 $\mathbb{Z}_2$ 的 GGSO 指数矩阵 $G_{ij} \in \{0,1\}$，并将式 (95) 和式 (98) 中相因子的指数 (模 2) 整合为

$$L\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} = a + b + ab + HG - \frac{1}{2}(\mathbf{a} - [\mathbf{a}])\cdot\mathbf{b}. \tag{104}$$

The evaluation of $L_{ij} \equiv L\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix}$ should be performed with special care, since modulo 2 periodicities do matter in the last term. The matrix $\mathbf{M}$ is then obtained (modulo 2) by

$L_{ij} \equiv L\begin{bmatrix}\beta_i\\ \beta_j\end{bmatrix}$ 的计算需要特别小心，因为最后一项中模 2 周期性的影响不可忽略。矩阵 $\mathbf{M}$ 可通过下式得到 (模 2)

$$\mathbf{M} = \mathbf{S}^{-1}(\mathbf{G} + \mathbf{L})\mathbf{S}^{-T}. \tag{105}$$

Carrying out these steps for the example model at hand, we obtain

对本文所用的示例模型执行上述步骤后，我们得到

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{106}$$

Plugging this matrix into (101), it is straightforward to extract the modular invariant phase Φ in terms of the summation parameters. The partition function (89) with Φ determined using this procedure then precisely reproduces Eq. (49) term by term.

将该矩阵代入式 (101)，可以很容易地从求和参数中提取出模不变相 Φ 。通过该流程确定了 Φ 后，配分函数 (89) 可以逐项精确重现式 (49)。

Up to this point, the map was clearly a bijective one. However, the partition function in Eq. (89) is not yet in the orbifold representation. After all, all we did so far was reorganize the terms of (49) into a useful intermediate form. The actual orbifold representation, however, is just around the corner. Notice that the orbifold partition function is not formulated in terms of γ_i, δ_i but should be expressed in terms of Narain lattices with twists and shifts, at the special loci in moduli space compatible with bosonization.

到目前为止，该映射显然是双射。但式 (89) 的配分函数还不是轨形表示，迄今为止我们做的工作只是将式 (49) 的项重新整理为方便的中间形式。而真正的轨形表示其实呼之欲出。注意轨形配分函数不是用 γ_i, δ_i 表述的，它应当用带有扭转和平移的 Narain 晶格表示，并且定义在与玻色化相容的模空间特殊轨迹上。

To this end, it is necessary to obtain a generalized version of (77) and (78) applicable to our case, where all six compactified coordinates are fermionized with different boundary conditions (γ_i, δ_i) . The contribution of the fermionized coordinates will be organized as a product of three lattices of signature $(2, 2)$, which will now contain both shifts and twists. For example, for the first lattice, we have

为此，需要得到适用于我们情形的 (77) 和 (78) 的推广形式，在我们的情形中所有六个紧致化坐标都通过不同的边界条件 (γ_i, δ_i) 费米化。费米化坐标的贡献会被整理为三个符号为 $(2, 2)$ 的格的乘积，其中现在同时包含平移和扭转。例如，对于第一个格，我们有

$$\Gamma_{2,2}\left[\begin{array}{ccc} H_1 & H_2 & h_1 \\ G_1 & G_2 & g_1 \end{array}\right](2\mathrm{i}\alpha',\mathrm{i}) = \frac{1}{4}\sum_{\substack{\gamma_1,\delta_1=0,1\\ \gamma_2,\delta_2=0,1}} (-1)^{\gamma_1 G_1+\delta_1 H_1+H_1 G_1}(-1)^{\gamma_2 G_2+\delta_2 H_2+H_2 G_2}$$
$$\times\left|\vartheta\left[\begin{array}{c}\gamma_1\\ \delta_1\end{array}\right]\vartheta\left[\begin{array}{c}\gamma_1+h_1\\ \delta_1+g_1\end{array}\right]\vartheta\left[\begin{array}{c}\gamma_2\\ \delta_2\end{array}\right]\vartheta\left[\begin{array}{c}\gamma_2+h_1\\ \delta_2+g_1\end{array}\right]\right|, \quad (107)$$

and similarly for the remaining two lattices. Here, (h_1, g_1) are again associated with the $\mathbb{Z}_2$ twist that rotates the coordinates of the first 2-torus. The new parameters (H_1, G_1) and (H_2, G_2) are similarly ascribed to additional $\mathbb{Z}_2$ orbifold factors which, respectively, act as translations (shifts) along the two cycles of the T^2 . The fermionization point now corresponds [18] to $T = 2\mathrm{i}\alpha', U = \mathrm{i}$, while for vanishing twist $h_1 = g_1 = 0$, one recovers the shifted lattice

其余两个格同理。此处，(h_1, g_1) 再次与转动第一个二维环面坐标的 $\mathbb{Z}_2$ 扭转相关联。新参数 (H_1, G_1) 和 (H_2, G_2) 同理被赋予额外的 $\mathbb{Z}_2$ 轨形因子，这些因子分别沿 T^2 的两个闭链做平移 (平移变换)。费米化点现在对应 [18] 于 $T = 2\mathrm{i}\alpha', U = \mathrm{i}$ ， 而当扭转 $h_1 = g_1 = 0$ 消失时，我们就得到平移格

$$\Gamma_{2,2}\left[\begin{array}{ccc} H_1 & H_2 & 0 \\ G_1 & G_2 & 0 \end{array}\right](T,U) = \sum_{m_i,n_i\in\mathbb{Z}} (-1)^{m_1 G_1+m_2 G_2} q^{\frac{\alpha'}{4}|P_L|^2}\bar{q}^{\frac{\alpha'}{4}|P_R|^2}, \quad (108)$$

where now the complexified momenta also depend on the shift parameters:

此时复化动量也依赖于平移参数:

(109)

$$P_L = \frac{m_2 - Um_1 + T\left(n_1 + \frac{H_1}{2} + U\left(n_2 + \frac{H_2}{2}\right)\right)}{\sqrt{T_2 U_2}},$$
$$P_R = \frac{m_2 - Um_1 + \bar{T}\left(n_1 + \frac{H_1}{2} + U\left(n_2 + \frac{H_2}{2}\right)\right)}{\sqrt{T_2 U_2}}.$$

It is easy to invert (107) and rewrite it in a more convenient form

我们很容易对 (107) 求逆，将其改写为更方便的形式

$$\frac{1}{4}\sum_{\substack{\gamma_1,\delta_1=0,1\\ \gamma_2,\delta_2=0,1}} (-1)^{\gamma_1 Y_1+\delta_1 X_1+\gamma_2 Y_2+\delta_2 X_2}\left|\vartheta\begin{bmatrix}\gamma_1\\ \delta_1\end{bmatrix}\vartheta\begin{bmatrix}\gamma_1+h_1\\ \delta_1+g_1\end{bmatrix}\vartheta\begin{bmatrix}\gamma_2\\ \delta_2\end{bmatrix}\vartheta\begin{bmatrix}\gamma_2+h_1\\ \delta_2+g_1\end{bmatrix}\right|$$
$$= \frac{1}{2^2}\sum_{\substack{H_1,G_1=0,1\\ H_2,G_2=0,1}} (-1)^{H_1 Y_1+G_1 X_1+X_1 Y_1}(-1)^{H_2 Y_2+G_2 X_2+X_2 Y_2}\Gamma_{2,2}\left[\begin{array}{ccc} H_1 & H_2 & h_1 \\ G_1 & G_2 & g_1 \end{array}\right](2\mathrm{i}\alpha',\mathrm{i}),$$

(110)

valid for any $\mathbb{Z}_2$ -valued parameters X_1, X_2, Y_1, Y_2 . This may be now used in order to replace the (γ_i, δ_i) -coupled theta functions with Narain lattices. Doing so, the partition function of the theory can finally be brought into its orbifold representation

该式对任意取值为 $\mathbb{Z}_2$ 的参数 X_1, X_2, Y_1, Y_2 都成立。现在可以用它将 (γ_i, δ_i) 耦合的 θ 函数替换为 Narain 格。通过这样做，该理论的配分函数最终可以写成它的轨形表示形式

$$
\begin{aligned}
Z = & \frac{1}{\eta^{12}\bar{\eta}^{24}} \frac{1}{2^{12}} \sum_{\{A'\},\{B'\}} (-1)^{a+b+ab+HG+\Phi'\begin{bmatrix}\{A'\}\\ \{B'\}\end{bmatrix}} \\
& \times \vartheta\begin{bmatrix} a \\ b \end{bmatrix} \vartheta\begin{bmatrix} a+h_1 \\ b+g_1 \end{bmatrix} \vartheta\begin{bmatrix} a+h_2 \\ b+g_2 \end{bmatrix} \vartheta\begin{bmatrix} a-h_1-h_2 \\ b-g_1-g_2 \end{bmatrix} \\
& \times \Gamma_{2,2}\begin{bmatrix} H_1 & H_2 & h_1 \\ G_1 & G_2 & g_1 \end{bmatrix} \Gamma_{2,2}\begin{bmatrix} H_3 & H_4 & h_2 \\ G_3 & G_4 & g_2 \end{bmatrix} \Gamma_{2,2}\begin{bmatrix} H_5 & H_6 & h_1+h_2 \\ G_5 & G_6 & g_1+g_2 \end{bmatrix} \\
& \times \bar{\vartheta}^{5}\begin{bmatrix} k \\ \ell \end{bmatrix} \bar{\vartheta}\begin{bmatrix} k+h_1 \\ \ell+g_1 \end{bmatrix} \bar{\vartheta}\begin{bmatrix} k+h_2 \\ \ell+g_2 \end{bmatrix} \bar{\vartheta}\begin{bmatrix} k-h_1-h_2 \\ \ell-g_1-g_2 \end{bmatrix} \times \bar{\vartheta}^{4}\begin{bmatrix} \rho \\ \sigma \end{bmatrix} \bar{\vartheta}^{4}\begin{bmatrix} \rho+H \\ \sigma+G \end{bmatrix},
\end{aligned}
$$

(111)

---

[18] The factorization point may appear to be in conflict with the value $T/\alpha' = U = \mathrm{i}$ obtained earlier in Eq. (77). However, the difference is that the lattice of Eq. (107) contains two independent shifts acting along the two T^2 directions and the factorization point is also sensitive to their embedding in the space of momenta and windings. The factorization point $T/2\alpha' = U = \mathrm{i}$ corresponds to both translations acting as momentum shifts.

[18] 因子化点看似与我们先前在式 (77) 中得到的取值 $T/\alpha' = U =$ 矛盾。但区别在于，式 (107) 的格包含两个作用于两个 T^2 方向的独立平移，而因子化点也依赖于它们在动量与绕数空间中的嵌入。因子化点 $T/2\alpha' = U = \mathrm{i}$ 对应两个平移都作为动量平移作用。

---

where $\{A'\} = \{a, k, \rho, \{H_i\}, h_1, h_2, H\}$ and $\{B'\} = \{b, \ell, \sigma, \{G_i\}, g_1, g_2, G\}$ and $i = 1, \ldots, 6$ are the new summation variables. Importantly, Φ' is the new modular invariant phase obtained from Φ under the phase substitution implied by (110). Concretely, one finds

其中 $\{A'\} = \{a, k, \rho, \{H_i\}, h_1, h_2, H\}$、$\{B'\} = \{b, \ell, \sigma, \{G_i\}, g_1, g_2, G\}$ 和 $i = 1, \ldots, 6$ 是新的求和变量。重要的是，Φ' 是由 (110) 隐含的相位替换后从 Φ 得到的新模不变相位。具体可得

$$
\begin{aligned}
\Phi' = & k\ell + [kg_1 + \ell h_1 + h_1 g_1] + [kg_2 + \ell h_2 + h_2 g_2] \\
& + [G_1 \rho + H_1 \sigma + H_1 G_1] + [G_2 (\rho + H) + H_2 (\sigma + G) + H_2 G_2] \\
& + [G_3 (k + H) + H_3 (\ell + G) + H_3 G_3] + [G_4 (k + \rho + H) + H_4 (\ell + \sigma + G)] \\
& + [G_5 (k + H) + H_5 (\ell + G) + H_5 G_5] + [G_6 (k + \rho + H) + H_6 (\ell + \sigma + G)]
\end{aligned}
$$

$$+\left[h_1 g_2 + g_1 h_2\right] + \left[h_1\left(G_3+G_4+G_5+G_6\right) + g_1\left(H_3+H_4+H_5+H_6\right)\right]$$

$$+\left[h_2\left(G_3+G_4+G_5+G_6\right) + g_2\left(H_3+H_4+H_5+H_6\right)\right]$$

$$+(14)+(23)+(34)+(46)+(35)+(56)\,,$$

(112)

where in the last line, the terms of the form (ij) stand for $G_iH_j+H_iG_j$, and we have grouped terms which are separately modular invariant into square brackets. In this representation, it is possible to recognize the orbifold action. Aside from the familiar $\mathbb{Z}_2\times\mathbb{Z}_2$ rotating the first and second pair of 2-tori respectively, the second, third, and fourth lines of (112) indicate the free actions $(-1)^{F_2}\sigma_1, (-1)^{F_2}\sigma_2, (-1)^{F_1}\sigma_3$, $(-1)^{F_1+F_2}\sigma_4, (-1)^{F_1}\sigma_5$ and $(-1)^{F_1+F_2}\sigma_6$, where σ_I denote the order-2 shifts along each direction and F_1, F_2 are the "fermion numbers" associated with the spinorial representations of the original E_8 factors, respectively. The remaining terms in the phases correspond to chirality conventions and discrete torsion phases.

最后一行中，形如 (ij) 的项代表 $G_iH_j+H_iG_j$ ，我们已经将各自模不变的项归入方括号中。在该表示下可以识别出 orbifold 作用。除了我们熟悉的分别转动两对 2 维环面的 $\mathbb{Z}_2\times\mathbb{Z}_2$ 之外，(112) 的第二、三、四行给出了自由作用 $(-1)^{F_2}\sigma_1, (-1)^{F_2}\sigma_2, (-1)^{F_1}\sigma_3$ 、$(-1)^{F_1+F_2}\sigma_4, (-1)^{F_1}\sigma_5$ 和 $(-1)^{F_1+F_2}\sigma_6$ ，其中 σ_I 代表沿各个方向的 2 阶平移，F_1, F_2 分别是与原 E_8 因子的旋量表示关联的“费米子数”。相位中的剩余项对应手征约定和离散挠率相位。

As a cross-check, it is instructive to compare the contributions of bosonic and fermionic states to the partition function both in the FFF and in the orbifold formulations using (49) and (111), respectively, at the fermionic point. One obtains

作为交叉检验，在费米点分别利用 (49) 和 (111)，对比 FFF 表述与 orbifold 表述中玻色态和费米态对配分函数的贡献是很有启发意义的。我们得到

$$Z_{\mathrm{B}} = -Z_{\mathrm{F}} = 2\bar{q}^{-1} + 872 + 120q^{1/2}\bar{q}^{-1/2} + 32q\bar{q}^{-1} + 4080q^{1/8}\bar{q}^{1/8}$$

$$+16q^{9/8}\bar{q}^{-7/8} + 704q^{5/8}\bar{q}^{-3/8} + \ldots\,, \tag{113}$$

and the contributions match, as they should. Note that the contribution of bosonic states Z_{B} exactly cancels that of the fermionic ones Z_{F} due to the unbroken spacetime supersymmetry of the theory. Note, furthermore, that each term comes with positive integer multiplicity, as required for a correct particle interpretation.

且贡献正如预期那样吻合。注意由于理论未破缺的时空超对称性，玻色态 Z_{B} 的贡献恰好抵消了费米态 Z_{F} 的贡献。此外还需注意，每一项都带有正整数简并度，符合正确粒子诠释的要求。

With the map from the FFF to the orbifold representation established, it is possible to reinstate the moduli dependence back into the Narain lattices and deform the theory away from the fermionic point. In particular, this is the natural path to take if one is interested in studying supersymmetry breaking in FFF models. For instance, in the context of the particular vacuum discussed here, it is easy to see that $\mathcal{N}=1$ supersymmetry is unbroken, as may be checked either by explicitly constructing the supercharges or by means of Jacobi identities. In non-supersymmetric constructions, one may extract and study the gravitino mass $m_{3/2}$ as a function of the compactification moduli and obtain conditions for the breaking to be spontaneous directly in terms of the GGSO coefficients. [19]

在建立了从 FFF 到 orbifold 表示的映射后，我们可以将模依赖重新引入 Narain 格，并将理论偏离费米点形变。特别地，当我们研究 FFF 模型中的超对称破缺时，这是一条自然的路径。例如，在本文讨论的这个特殊真空中，很容易看出 $\mathcal{N}=1$ 超对称未发生破缺，这可以通过显式构造超荷，也可以通过雅可比恒等式验证。在非超对称构造中，我们可以提取并研究作为紧致化模函数的引力微子质量 $m_{3/2}$，并直接用 GGSO 系数得到自发破缺的条件。[19]

It is also important to mention that the orbifold representation in terms of the phase (112) is not unique. There are equivalent representations related by lattice redefinitions and T-dualities, arising from the different ways of performing the summations over γ_i, δ_i . We shall not elaborate on this here, but instead refer the interested reader to [98] for a detailed discussion.

还需要指出，(112) 中用相位表示的 orbifold 表示并不是唯一的。等价表示通过格重新定义和 T 对偶联系起来，源于对 γ_i, δ_i 求和的不同方式。我们在此不展开讨论，感兴趣的读者可以参阅文献 [98] 获取详细讨论。

Cross-References

交叉引用

- A Lightning Introduction to String Theory

- 弦理论闪电简介

Effective Field Theory for Compact Binary Dynamics

致密双星动力学的有效场论

Heterotic Orbifold Models

杂化轨形模型

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[19] For a discussion, see [96].

[19] 相关讨论参见文献 [96]。

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